

PROBLEM 1: Examine if the following statements are true or false with correct reasoning. Random guessing or incorrect reasoning fetches zero credit. A statement is true if it is generic for all cases. A counter example is enough to make it false. All sub-parts of this problem carry equal credit.

- (1) Let  $\lambda$  and  $x$  be an eigen value and the corresponding eigen vector of a square matrix  $A := [a_{ij}]$  respectively. Then,  $|\lambda| \geq \|A\|$ . The norm is the Hilbert Schmidt norm i.e.,  $\sqrt{\sum_i \sum_j |a_{ij}|^2}$ .
- (2) Let  $X$  and  $Y$  be any two random variables related by  $Y = \phi(X)$  where,  $\phi(\cdot)$  is a monotonically increasing function. The probability distribution functions of the random variables are related as  $F_Y(\phi(x)) \geq F_X(x)$ .
- (3) The energy of a signal  $s(t)$  defined over  $[0, T]$  is  $E = \int_0^T s^2(t) dt$ . Let  $\bar{s}$  be the signal space representation of  $s(t)$  over an appropriate basis. Then,  $E = \|\bar{s}\|$  where  $\|\cdot\|$  is the usual  $\mathcal{L}_2$  norm.
- (4) Allpass filters are always causal and stable.
- (5) We need  $2n$  modes and  $2^n$  measurements to determine the frequencies from a noise free linear mixture having  $n$  sinusoids.

(25 pts.)

1) FALSE.

Consider  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  whose eigenvalues are  $\lambda_1 = \lambda_2 = 1$ . But  $\|A\| = \sqrt{2}$

2) FALSE

$$\begin{aligned} F_Y(\phi(x)) &= P_Y[Y \leq \phi(x)] = P_Y[\phi(X) \leq \phi(x)] \\ &= P_Y[X \leq x] \quad (\text{Since } \phi(\cdot) \text{ is monotonically increasing}) \\ \therefore F_Y(\phi(x)) &= F_X(x) \end{aligned}$$

3) FALSE

$$\mathcal{L}_2 \text{ norm is } \|s\| = \sqrt{\int_0^T s^2(t) dt} = \sqrt{E}$$

4) FALSE

$H(z) = z$  is all-pass but not causal.

5) FALSE

We need  $2n$  modes and  $4n$  measurements.

PROBLEM 2: This problem has 2 parts.

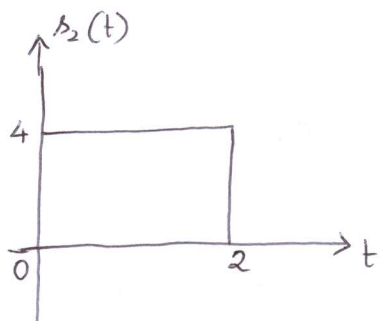
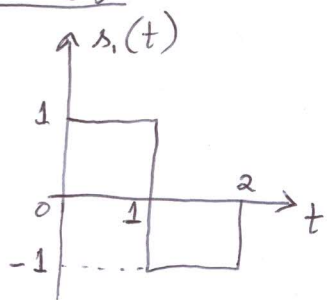
- (1) Suppose  $\{\phi_i\}_{i=1}^n$  be a set of orthonormal basis for a vector space  $\mathcal{V}$ . Prove that for any two vectors  $x$  and  $y$  belonging to  $\mathcal{V}$ ,  $\langle x, y \rangle = \sum_{i=1}^n \langle x, \phi_i \rangle \overline{\langle y, \phi_i \rangle}$ . (7 pts.)
- (2) A source emits two signals  $s_1(t) = u(t) - 2u(t-1) + u(t-2)$  and  $s_2(t) = 4u(t) - 4u(t-2)$  where,  $u(t)$  is the standard unit step function with probabilities  $p$  and  $1-p$  respectively. Represent the signals as vectors in an appropriate coordinate system. Let us suppose that the signal vectors are to be transmitted over an AWGN channel  $\mathcal{N}(0, \sigma^2)$  where the noise cloud acts independently on each of the signal coordinates. Obtain the optimal linear decision boundaries for this system. Evaluate the probability of error. (18 pts.)

Part (1):

$$y = \sum_{i=1}^n \langle y, \phi_i \rangle \phi_i$$

$$\begin{aligned} \Rightarrow \langle x, y \rangle &= \left\langle x, \sum_{i=1}^n \langle y, \phi_i \rangle \phi_i \right\rangle = \sum_{i=1}^n \langle x, \langle y, \phi_i \rangle \phi_i \rangle \\ &= \sum_{i=1}^n \langle x, \phi_i \rangle \overline{\langle y, \phi_i \rangle} \quad (\because \langle x, ay \rangle = \bar{a} \langle x, y \rangle) \end{aligned}$$

Part (2):



$$\|s_1(t)\|^2 = \int_0^2 1 dt = 2$$

$$\|s_2(t)\|^2 = \int_0^2 4^2 dt = 32$$

$\langle s_1(t), s_2(t) \rangle = 0$ .  $\therefore$  We can choose the orthonormal basis as

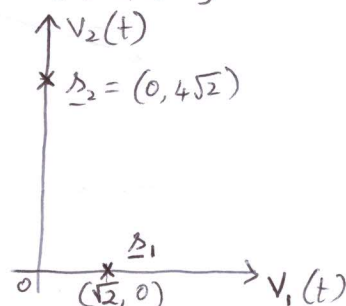
$$v_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \frac{1}{\sqrt{2}} s_1(t)$$

$$v_2(t) = \frac{s_2(t)}{\|s_2(t)\|} = \frac{1}{4\sqrt{2}} s_2(t)$$

In the chosen orthonormal basis, the signals are represented as  $\underline{s}_1 = (\sqrt{2}, 0)$  and  $\underline{s}_2 = (0, 4\sqrt{2})$

The received signal is

$$\underline{y} = \begin{cases} \underline{s}_1 + (n_1, n_2) & \text{w.p. } (p) \\ \underline{s}_2 + (n_1, n_2) & \text{w.p. } (1-p) \end{cases} \quad n_1, n_2 \sim \mathcal{N}(0, \sigma^2)$$



The likelihood probability densities are

$$P(\underline{x} | \underline{\Delta}_1) = P((n_1, n_2) = \underline{x} - \underline{\Delta}_1) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} \|\underline{x} - \underline{\Delta}_1\|^2\right\} \quad \text{--- (1)}$$

Similarly,  $P(\underline{x} | \underline{\Delta}_2) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} \|\underline{x} - \underline{\Delta}_2\|^2\right\} \quad \text{--- (2)}$

The a posteriori probabilities are

$$P(\underline{\Delta}_1 | \underline{x}) = P(\underline{x} | \underline{\Delta}_1) \frac{P_x[\underline{\Delta}_1]}{P(\underline{x})} \quad \forall \underline{x} \in \mathbb{R}^2 \quad \text{since } P(\underline{x}) \neq 0 \quad \forall \underline{x} \in \mathbb{R}^2 \quad \text{--- (3)}$$

$$P(\underline{\Delta}_2 | \underline{x}) = P(\underline{x} | \underline{\Delta}_2) \frac{P_x[\underline{\Delta}_2]}{P(\underline{x})} \quad \forall \underline{x} \in \mathbb{R}^2 \quad \text{--- (4)}$$

The optimal decision boundary is given by

$$P(\underline{\Delta}_1 | \underline{x}) = P(\underline{\Delta}_2 | \underline{x}) \quad \left(\text{Using (3) \& (4)}\right)$$

$$\Rightarrow P(\underline{x} | \underline{\Delta}_1) P_x[\underline{\Delta}_1] = P(\underline{x} | \underline{\Delta}_2) P_x[\underline{\Delta}_2]$$

$$\Rightarrow \exp\left\{\frac{\|\underline{x} - \underline{\Delta}_2\|^2 - \|\underline{x} - \underline{\Delta}_1\|^2}{2\sigma^2}\right\} = \frac{1-P}{P}$$

$$\Rightarrow \langle \underline{x}, (\underline{\Delta}_1 - \underline{\Delta}_2) \rangle + \|\underline{\Delta}_2\|^2 - \|\underline{\Delta}_1\|^2 = 2\sigma^2 \ln\left(\frac{1-P}{P}\right)$$

Using  $\underline{x} = (x, y)$ , the boundary is given by

$$\sqrt{2}x - 4\sqrt{2}y + 24 + 2\sigma^2 \ln\left(\frac{P}{1-P}\right) = 0$$

