PROBLEM 1: Examine if the following statements are true or false with correct reasoning. Random guessing or incorrect reasoning fetches zero credit. A statement is true if it is generic for all cases. A counter example is enough to make it false. All sub-parts of this problem carry equal credit.

(1) Let λ and x be an eigen value and the corresponding eigen vector of a square matrix $A := [a_{ij}]$ respectively. Then, $|\lambda| \geq ||A||$. The norm is the Hilbert Schmidt norm i.e., $\sqrt{\sum_i \sum_j |a_{ij}|^2}$.

(2) Let X and Y be any two random variables related by $Y = \phi(X)$ where, $\phi(.)$ is a monotonically increasing function. The probability distribution functions of the random variables are related as $F_Y(\phi(x)) \geq F_X(x)$.

(3) The energy of a signal $s(t)$ defined over $[0, T]$ is $E = \int_0^T s^2(t)dt$. Let $\bar{s}$ be the signal space representation of $s(t)$ over an appropriate basis. Then, $E = ||s||$ where $||.||$ is the usual $\mathcal{L}_2$ norm.

(4) Allpass filters are always causal and stable.

(5) We need $2n$ modes and 2^n measurements to determine the frequencies from a noise free linear mixture having n sinusoids.

(25 pts.)

1) FALSE.

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ whose eigenvalues are $\lambda_1 = \lambda_2 = 1$. But $||A|| = \sqrt{2}$

2) FALSE

$F_Y(\phi(x)) = P_r[Y \leq \phi(x)] = P_r[\phi(X) \leq \phi(x)]$

$= P_r[X \leq x]$ (Since $\phi(\cdot)$ is monotonically increasing)

$\therefore F_Y(\phi(x)) = F_X(x)$

3) FALSE

$\mathcal{L}_2$ norm is $||S|| = \sqrt{\int_0^T s^2(t)\,dt} = \sqrt{E}$

4) FALSE

$H(z) = z$ is all-pass but not causal.

5) FALSE

We need $2n$ modes and $4n$ measurements.

PROBLEM 2: This problem has 2 parts.

(1) Suppose $\{\phi_i\}_{i=1}^{n}$ be a set of orthonormal basis for a vector space $\mathcal{V}$. Prove that for any two vectors x and y belonging to $\mathcal{V}$, $\langle x, y\rangle = \sum_{i=1}^{n} \langle x, \phi_i\rangle \overline{\langle y, \phi_i\rangle}$. (7 pts.)

(2) A source emits two signals $s_1(t) = u(t) - 2u(t-1) + u(t-2)$ and $s_2(t) = 4u(t) - 4u(t-2)$ where, $u(t)$ is the standard unit step function with probabilities p and $1-p$ respectively. Represent the signals as vectors in an appropriate coordinate system. Let us suppose that the signal vectors are to be transmitted over an AWGN channel $\mathcal{N}(0, \sigma^2)$ where the noise cloud acts independently on each of the signal coordinates. Obtain the optimal linear decision boundaries for this system. Evaluate the probability of error. (18 pts.)

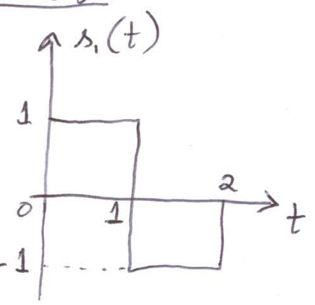
Part (1):

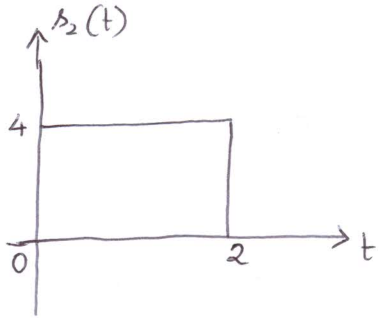
$$y = \sum_{i=1}^{n} \langle y, \phi_i\rangle \phi_i$$

$$\Rightarrow \langle x, y\rangle = \langle x, \sum_{i=1}^{n} \langle y, \phi_i\rangle \phi_i\rangle = \sum_{i=1}^{n} \langle x, (\langle y, \phi_i\rangle \phi_i)\rangle$$

$$= \sum_{i=1}^{n} \langle x, \phi_i\rangle \overline{\langle y, \phi_i\rangle} \quad \left(\because \langle x, ay\rangle = \bar{a}\langle x, y\rangle\right)$$

Part (2):





$$\|s_1(t)\|^2 = \int_0^2 1\,dt = 2$$

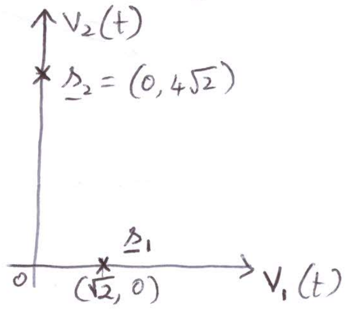
$$\|s_2(t)\|^2 = \int_0^2 4^2\,dt = 32$$

$\langle s_1(t), s_2(t)\rangle = 0 \quad \therefore$ We can chose the orthonormal basis as

$$V_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \frac{1}{\sqrt{2}} s_1(t)$$

$$V_2(t) = \frac{s_2(t)}{\|s_2(t)\|} = \frac{1}{4\sqrt{2}} s_2(t)$$

In the chosen orthonormal basis, the signals are represented as $\underline{s}_1 = (\sqrt{2}, 0)$ and $\underline{s}_2 = (0, 4\sqrt{2})$



The received signal is

$$\underline{r} = \begin{cases} \underline{s}_1 + (n_1, n_2) & \text{w.p. } (p) \\ \underline{s}_2 + (n_1, n_2) & \text{w.p. } (1-p) \end{cases} \qquad n_1, n_2 \sim \mathcal{N}(0, \sigma^2)$$

The likelihood probability densities are

$$p(\underline{r} \mid \underline{s}_1) = p((n_1, n_2) = \underline{r} - \underline{s}_1) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} \|\underline{r} - \underline{s}_1\|^2\right\} \quad \text{— (1)}$$

Similarly, $$p(\underline{r} \mid \underline{s}_2) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} \|\underline{r} - \underline{s}_2\|^2\right\} \quad \text{— (2)}$$

The aposterior probabilities are

$$p(\underline{s}_1 \mid \underline{r}) = \frac{p(\underline{r} \mid \underline{s}_1)\, P_r[\underline{s}_1]}{p(\underline{r})} \quad \forall\, \underline{r} \in \mathbb{R}^2 \text{ since } p(\underline{r}) \neq 0 \ \forall\, \underline{r} \in \mathbb{R}^2 \quad \text{— (3)}$$

$$p(\underline{s}_2 \mid \underline{r}) = \frac{p(\underline{r} \mid \underline{s}_2)\, P_r[\underline{s}_2]}{p(\underline{r})} \quad \forall\, \underline{r} \in \mathbb{R}^2 \quad \text{— (4)}$$

The optimal decision boundary is given by

$$p(\underline{s}_1 \mid \underline{r}) = p(\underline{s}_2 \mid \underline{r})$$

(Using (3) & (4))

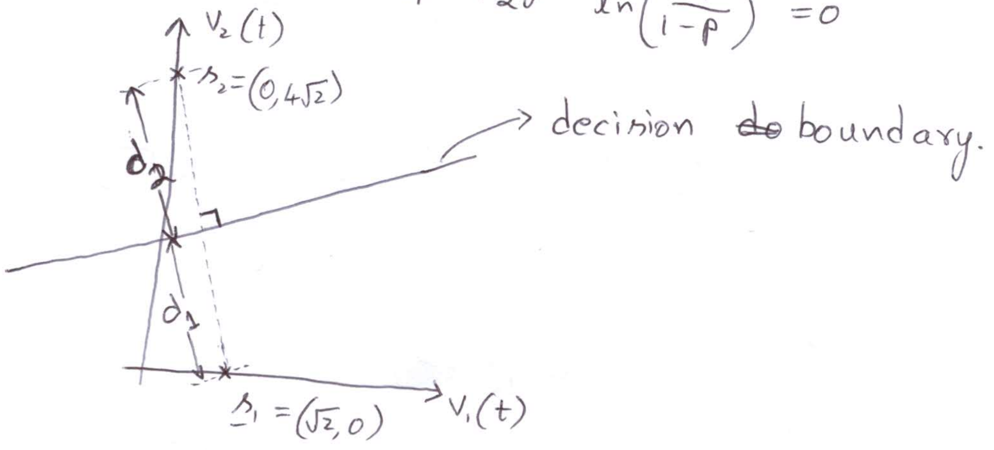
$$\Rightarrow p(\underline{r} \mid \underline{s}_1)\, P_r[\underline{s}_1] = p(\underline{r} \mid \underline{s}_2)\, P_r[\underline{s}_2]$$

$$\Rightarrow \exp\left\{\frac{\|\underline{r} - \underline{s}_2\|^2 - \|\underline{r} - \underline{s}_1\|^2}{2\sigma^2}\right\} = \frac{1-p}{p}$$

$$\Rightarrow \langle \underline{r}, (\underline{s}_1 - \underline{s}_2) \rangle + \|\underline{s}_2\|^2 - \|\underline{s}_1\|^2 = 2\sigma^2 \ln\left(\frac{1-p}{p}\right)$$

Using $\underline{r} = (x, y)$, the boundary is given by

$$\sqrt{2}\, x - 4\sqrt{2}\, y + 24 + 2\sigma^2 \ln\left(\frac{p}{1-p}\right) = 0$$



The optimal decision region for s_1 is

$$R_1 = \left\{ (x,y) \,\middle|\, \sqrt{2}\,x - 4\sqrt{2}\,y + 24 + 2\sigma^2 \ln\left(\frac{p}{1-p}\right) \geq 0 \right\}$$

for s_2, the optimal decisior ~~redion~~ region is

$$R_2 = \left\{ (x,y) \,\middle|\, \sqrt{2}\,x - 4\sqrt{2}\,y + 24 + 2\sigma^2 \ln\left(\frac{p}{1-p}\right) < 0 \right\}$$

Let s_t represent the transmitted signal and s_d represent the decision made based on received signal.

The probability of detection error is

$$P_e = \Pr[s_t \neq s_d] = \Pr[s_t \neq s_d \mid s_t = s_1]\Pr[s_t = s_1] + \Pr[s_t \neq s_d \mid s_t = s_2]\Pr[s_t = s_2]$$

$$= \Pr[\underline{r} \in R_2 \mid s_t = s_1]\, p + \Pr[\underline{r} \in R_1 \mid s_t = s_2]\,(1-p)$$

$$= p \iint_{R_2} \frac{1}{2\pi\sigma^2} \exp\left\{ \frac{-1}{2\sigma^2}\left[(x-\sqrt{2})^2 + y^2\right] \right\} dx\,dy$$

$$+ (1-p) \iint_{R_1} \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{2\sigma^2}\left(x^2 + (y-4\sqrt{2})^2\right) \right\} dx\,dy$$

$$= p\, Q\left(\frac{d_1}{\sigma}\right) + (1-p)\, Q\left(\frac{d_2}{\sigma}\right)$$

Area of Gaussian pulse in a half plane can be computed as follows.
a) Rotate axis so that the boundary of half plane is the new X-axis.
b) The Gaussian function is now separable into Gaussian functions in X & Y directions.
c) Integral in X direction = 1
d) Integral in Y direction is given by Q(·) function.

where $Q(x) = \int_x^{\infty} e^{-y^2/2}\, dy$ and d_1 & d_2 are lengths of Perpendiculars ~~to~~ from s_1 & s_2 to decision boundary.

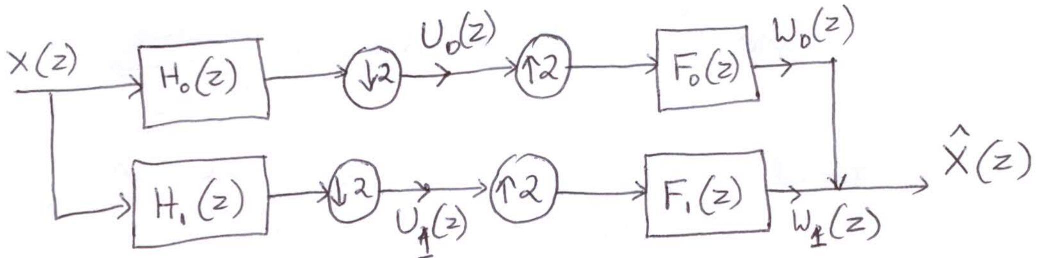
$$d_1 = \frac{\left|26 + 2\sigma^2 \ln\left(\frac{p}{1-p}\right)\right|}{\sqrt{34}} \quad \& \quad d_2 = \frac{\left|2\sigma^2 \ln\left(\frac{p}{1-p}\right) - 8\right|}{\sqrt{34}}$$

PROBLEM 3: This problem has 2 parts.

(1) Consider a 2 channel QMF bank with analysis and synthesis filters. Suppose $H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}$, and $H_1(z) = H_0(-z)$. Obtain a set of stable synthesis filters for perfect reconstruction. What can you comment on amplitude and phase distortion of this filter bank? (15 pts.)

(2) Consider a 3 channel filter bank. Suppose we downsample by 2 and upsample by 2 in first branch. In the subsequent 2 branches, suppose we downsample and upsample by 4. Assume that the pass band width of the analysis filters is compliant with the decimation rate that follows it. From first principles, obtain the expression for the output signal after synthesis. Investigate if we can get alias free reconstruction of the signal after synthesis. What are the necessary conditions for the same? (20 pts.)

Part (1):



$$U_0(z) = \frac{1}{2}\left(X(z^{1/2})H_0(z^{1/2}) + X(-z^{1/2})H_0(-z^{1/2})\right)$$

$$U_1(z) = \frac{1}{2}\left[X(z^{1/2})H_1(z^{1/2}) + X(-z^{1/2})H_1(-z^{1/2})\right]$$

$$W_0(z) = F_0(z)U_0(z^2) = \frac{1}{2}\left[X(z)H_0(z) + X(-z)H_0(-z)\right]F_0(z)$$

$$W_1(z) = F_1(z)U_1(z^2) = \frac{1}{2}\left[X(z)H_1(z) + X(-z)H_1(-z)\right]F_1(z)$$

Using $H_1(z) = H_0(-z)$ and $\hat{X}(z) = W_0(z) + W_1(z)$,

$$\hat{X}(z) = \frac{1}{2}X(z)\left[H_0(z)F_0(z) + H_0(-z)F_1(z)\right]$$
$$+ \frac{1}{2}X(-z)\left[H_0(-z)F_0(z) + H_0(z)F_1(z)\right]$$

For alias free output, we require $\dfrac{F_0(z)}{F_1(z)} = -\dfrac{H_0(z)}{H_0(-z)}$

Let $F_0(z) = \cancel{2}T(z)H_0(z)$ and $F_1(z) = -\cancel{2}T(z)H_0(-z)$

$$\Rightarrow \frac{\hat{X}(z)}{X(z)} = \frac{1}{2}T(z)\left[(H_0(z))^2 - (H_0(-z))^2\right] \cancel{= T(z)}$$

$$= T(z)\left(6z^{-1} + 5z^{-3} + z^{-5}\right)\left(\cancel{4}\,2 + z^{-2}\right)$$

$$= T(z)\,z^{-1}\left(3 + z^{-2}\right)\left(2 + z^{-2}\right)^2$$

Choosing $T(z) = \dfrac{1}{(3+z^{-2})(2+z^{-2})^2}$ gives $\dfrac{\hat{X}(z)}{X(z)} = z^{-1}$

$\therefore$ The synthesis filter banks are

$$F_0(z) = \frac{2+6z^{-1}+5z^{-3}+z^{-5}+z^{2}}{(3+z^{-2})(2+z^{-2})^2} \quad \& \quad F_1(z) = -\frac{2-6z^{-1}+z^{2}-5z^{-3}-z^{-5}}{(3+z^{-2})(2+z^{-2})^2}$$

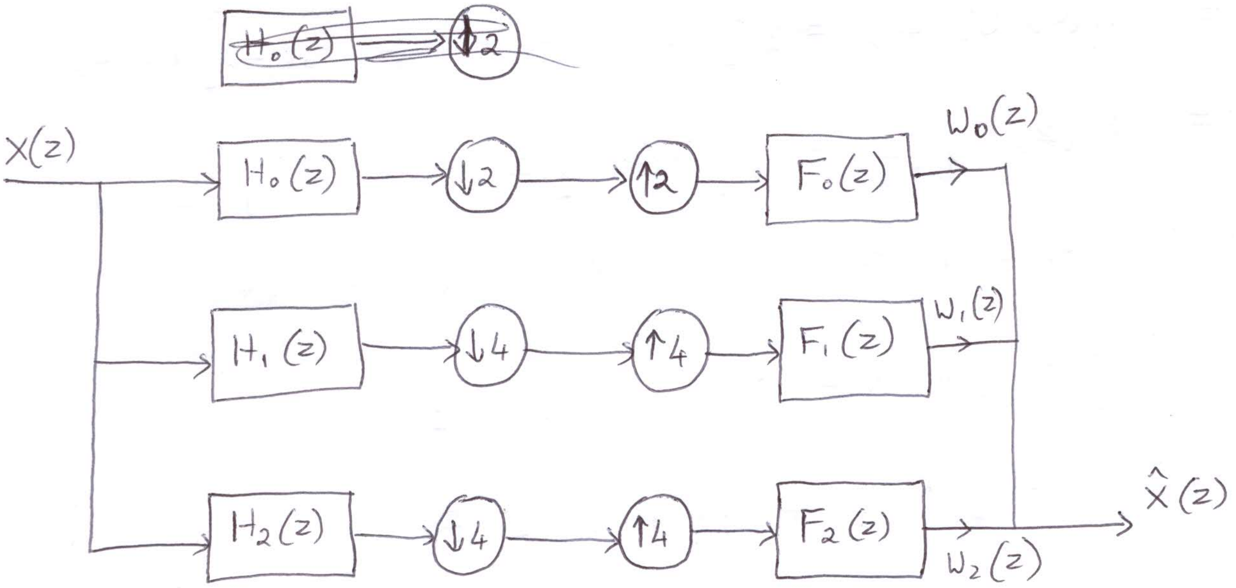
$F_0(z)$ & $F_1(z)$ has poles at $\frac{1}{\sqrt{3}}$ & $\frac{1}{\sqrt{2}}$.

$\therefore$ $F_0(z)$ & $F_1(z)$ are causal & stable.

The transfer function of the filter bank is $\dfrac{\hat{X}(z)}{X(z)} = z^{-1}$

$\therefore$ The filter bank does not have amplitude distortion but has linear phase distortion.

Part (2)



Let $j = \sqrt{-1}$. Then $1, j, -1, -j$ are 4^{th} roots of 1.

$$W_0(z) = \frac{1}{2} F_0(z)\left[X(z)H_0(z) + X(-z)H_0(-z)\right]$$

$$W_1(z) = \frac{1}{4} F_1(z)\left[X(z)H_1(z) + X(jz)H_1(jz) + X(-z)H_1(-z) + X(-jz)H_1(-jz)\right]$$

In matrix form,

$$\hat{X}(z) = \frac{1}{4}\begin{bmatrix} X(z) & X(jz) & X(-z) & X(-jz) \end{bmatrix} \begin{bmatrix} 2H_0(z) & H_1(z) & H_2(z) \\ 0 & H_1(jz) & H_2(jz) \\ 2H_0(-z) & H_1(-z) & H_2(-z) \\ 0 & H_1(-jz) & H_2(-jz) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}$$

For perfect reconstruction, we require

$$\begin{bmatrix} 2H_0(z) & H_1(z) & H_2(z) \\ 0 & H_1(jz) & H_2(jz) \\ 2H_0(-z) & H_1(-z) & H_2(-z) \\ 0 & H_1(-jz) & H_2(-jz) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- } (1)$$

To achieve $\begin{bmatrix} 0 & H_1(jz) & H_2(jz) \\ 2H_0(-z) & H_1(-z) & H_2(-z) \\ 0 & H_1(-jz) & H_2(-jz) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

the matrix must not have full rank. Therefore the rows 1 & 3 must be linearly dependent.

$$\therefore \frac{H_1(jz)}{H_1(-jz)} = \frac{H_2(jz)}{H_2(-jz)} \quad \text{--- } (2)$$

Using (2), (1) becomes

$$\underbrace{\begin{bmatrix} 2H_0(z) & H_1(z) & H_2(z) \\ 0 & H_1(jz) & H_2(jz) \\ 2H_0(-z) & H_1(-z) & H_2(-z) \end{bmatrix}}_{\mathbf{H}(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}}_{\mathbf{F}(z)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- } (3)$$

To achieve (3), rows 1 & 3 of $\mathbf{H}(z)$ must be linearly independend.

It is obvious that row 2 is linearly independent of rows 1 & 3.

∴ The necessary & sufficient conditions for perfect reconstruction are

a) $\mathbf{H}(z)$ must be full rank.

b) Condition (2) must hold true.

PROBLEM 4: Consider the equiripple FIR design. Let the signal be bandlimited to ω (π corresponds to Nyquist on the scale). Suppose we are interested in 3 separate operations (1) downsampling by M, (2) upsampling by L, (3) rational sampling rate conversion $\frac{L}{M}$.

(1) How would you choose the pass band edge frequencies of your filter? (6 pts.)

(2) How would you choose the stop band edge frequencies of your filter when (a) no aliasing is allowed (b) aliasing is permitted in the transition region of the decimator or beyond the Nyquist point in the interpolator as relevant to the three operations mentioned in this problem? (9 pts.)

Signal bandlimited to ω_0

Guiding principles for the choice of cutoff frequencies:

Pass band frequency (ω_p)

① Should be atleast signal bandwidth.

② Should be as low as possible for large transition band to reduce filter order.

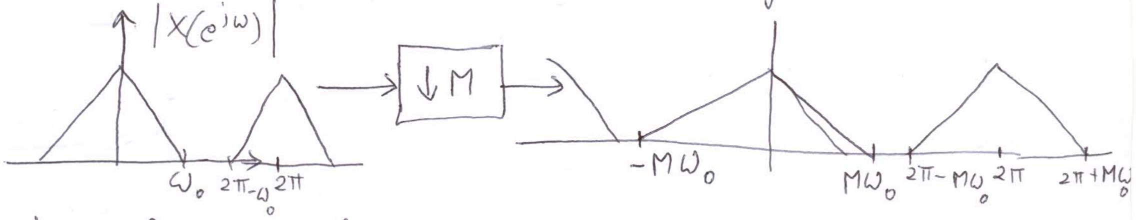
② Stop band frequency (ω_s)

① Should be less than the starting frequeny of undesired signal.

② Should be as large as possible for large transition band to reduce filter order.

Decimation Filter: $X(z) \rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow M} \rightarrow Y(z)$

In frequency domain, decimation stretches the signal by factor M & has replicas of the stretched signal separated by 2π



To avoid alias free-signal we require $M\omega_p \leq \pi \Rightarrow \omega_p \leq \frac{\pi}{M}$

$\therefore$ Pass band frequency $\omega_p = \min\left(\omega_0, \frac{\pi}{M}\right)$

a) No aliasing allowed: ~~Fisl~~ First undesired signal starts at $2\pi - M\omega_0$

we want $M\omega_s \leq 2\pi - M\omega_0 \Rightarrow \omega_s \leq \frac{2\pi}{M} - \omega_0$