

PROBLEM 1: This problem has 2 parts. Let $\phi(t) = \sum_k c_k \phi(2t - k)$ where $\phi(t)$ is the usual Haar scaling function over $[0, 1]$, $\{\phi(2t - k)\}$ are an orthonormal basis in \mathcal{V}_{-1} with $\mathcal{V}_0 \subset \mathcal{V}_{-1}$ and c_k are the corresponding Haar coefficients.

(1) Prove that $\sum_k c_k = 2$. (10 pts.)

(2) Let $\phi(\omega)$ be the continuous time Fourier transform of $\phi(t)$. Prove that $\phi(\omega) = \frac{1}{2} \left(\sum_k c_k e^{-\frac{j\omega k}{2}} \right) \phi\left(\frac{\omega}{2}\right)$. (10 pts.)

$$1) \quad \int \phi(t) dt = \int \sum_k c_k \phi(2t - k) dt$$

$$1 = \sum_k c_k \int \phi(2t - k) dt \quad (\because \text{finite } \Sigma \text{ over 'k'})$$

$$\text{Let } t' = 2t - k \quad dt' = 2 dt$$

$$1 = \sum_k \frac{c_k}{2} \underbrace{\int \phi(t') dt'}_1$$

$$\Rightarrow \sum_k c_k = 2$$

$$2) \quad F(\phi(t)) = \sum_k c_k F(\phi(2t - k)) \quad \text{by linearity of FT}$$

By scaling and translation properties

$$F(\phi(t)) = \sum_k c_k \frac{e^{-j\omega k}}{2} \phi\left(\frac{\omega}{2}\right)$$

NOTE:

$$F(\phi(2t - k)) = F\left(\phi\left(\underbrace{2}_{\text{scale}}\left(t - \underbrace{k/2}_{\text{translation}}\right)\right)\right)$$

PROBLEM 2: This problem has 2 parts.

(1) Obtain the Haar wavelet decomposition of the signal $s(t) = t^3$ over the interval $[0, 1]$ using the functions $\phi(t)$, $\psi(t)$, $\psi(2t)$ and $\psi(2t-1)$. The functions $\phi(t)$ and $\psi(t)$ are the usual Haar scaling and wavelet functions over the interval $[0, 1]$. (15 pts.)

(2) With the usual notations as followed in the class, prove that $V_j = V_J \oplus \bigoplus_{k=0}^{J-j-1} W_{J-k}$. You must explain all the steps throughout the proof carefully. (15 pts.)

$$1) \quad s(t) = c_0^{(0)} \phi(t) + d_0^{(0)} \psi(t) + d_0^{(-1)} \psi(2t) + d_1^{(-1)} \psi(2t-1)$$

$$c_0^{(0)} = \langle t^3, \phi(t) \rangle = \int_0^1 t^3 dt = \frac{1}{4}$$

$$d_0^{(0)} = \langle t^3, \psi(t) \rangle = \int_{1/2}^1 t^3 dt - \int_0^{1/2} t^3 dt = -\frac{7}{32}$$

$$d_0^{(-1)} = \langle t^3, \psi(2t) \rangle = \int_0^{1/4} t^3 dt - \int_{1/4}^{1/2} t^3 dt = -\frac{7}{512}$$

$$d_1^{(-1)} = \langle t^3, \psi(2t-1) \rangle = \int_{1/2}^{3/4} t^3 dt - \int_{3/4}^1 t^3 dt = -\frac{55}{512}$$

$$2) \quad v_{j-1} = v_j \oplus w_j \quad (\text{direct sum prop})$$

$$v_j = V_J \oplus W_J \oplus W_{J-1} \oplus \dots \oplus W_{j+1}$$

$$v_j = V_J \oplus \bigoplus_{k=0}^{J-j-1} W_{J-k}$$

(repeated application by decomposition)

PROBLEM 3: Suppose you are given a color image of size $N \times N$ specified in terms of R, G and B attributes. Note that R, G, B are red, green and blue colors used for describing a pixel within the image as in Figure 1. You can conveniently ignore the intensity attribute throughout this problem. Devise an algorithm for obtaining a *monochromatic* i.e., single color image from the color image using an appropriate linear transformation method. You need to describe the algorithmic steps clearly i.e., specifying the inputs, the procedure and outputs. You must indicate the dimensions of all the vectors and matrices while describing your procedure.

NOTE: You can make reasonable assumptions on any of the variables needed for the algorithm, but must clearly state them. (25 pts.)

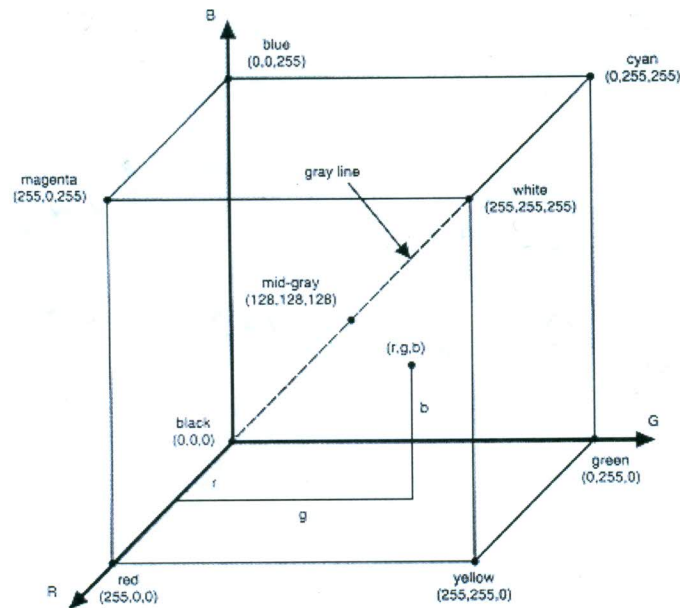


FIGURE 1. Color representation using R, G, B basis.

Inputs : $I_{N \times N}$, totally N^2 pixels.
Each pixel $I : [p_{ij}]$ is a vector with 3 coordinates.

Step 1 : Form $A = \begin{bmatrix} v_1 \\ \vdots \\ v_{N \times N} \end{bmatrix}_{N^2 \times 3}$

Step 2 : Calculate $\mu(A)$ and $\text{cov}(A)_{3 \times 3}$
 1×3

Step 3 : Sort the eigen values and eigen vectors corresponding to $\text{cov}(A)$.

Step 4 : Project each point v_i in the direction of the eigen vectors. Corresponding to \bar{r}, \bar{g} & \bar{b}
 $v_i' = \langle v_i, \hat{e} \rangle \hat{e} \quad 1 \times 3$

