PROBLEM 1: This problem has 2 parts. Let $\phi(t) = \sum_k c_k \phi(2t-k)$ where $\phi(t)$ is the usual Haar scaling function over [0,1], $\{\phi(2t-k)\}$ are an orthonormal basis in $\mathcal{V}_{-1}$ with $\mathcal{V}_0 \subset \mathcal{V}_{-1}$ and c_k are the corresponding Haar coefficients.

(1) Prove that $\sum_k c_k = 2$. (10 pts.)

(2) Let $\phi(\omega)$ be the continuous time Fourier transform of $\phi(t)$. Prove that $\phi(\omega) = \frac{1}{2}\left(\sum_k c_k e^{-\frac{j\omega k}{2}}\right)\phi(\frac{\omega}{2})$. (10 pts.)

1)
$$\int \phi(t)\,dt = \int \sum_k c_k \phi(2t-k)\,dt$$

$$1 = \sum_k c_k \int \phi(2t-k)\,dt \quad (\because \text{finite } \Sigma \text{ over 'k'}$$

Let $t' = 2t - k \qquad dt' = 2\,dt$

$$1 = \sum_k \frac{c_k}{2} \underbrace{\int \phi(t')\,dt'}_{1}$$

$$\Rightarrow \sum_k c_k = 2$$

2)
$$F(\phi(t)) = \sum_k c_k F(\phi(2t-k)) \quad \text{by linearity of FT}$$

By scaling and translation properties

$$F(\phi(t)) = \sum_k c_k \frac{e^{-j\omega k}}{2} \phi(\omega/2)$$

NOTE:

$$F(\phi(2t-k)) = F(\phi(\underset{\text{scale}}{2}(t - \underset{\text{translation}}{k/2})))$$

PROBLEM 2: This problem has 2 parts.

(1) Obtain the Haar wavelet decomposition of the signal $s(t) = t^3$ over the interval [0, 1] using the functions $\phi(t)$, $\psi(t)$, $\psi(2t)$ and $\psi(2t-1)$. The functions $\phi(t)$ and $\psi(t)$ are the usual Haar scaling and wavelet functions over the interval [0,1]. (15 pts.)

(2) With the usual notations as followed in the class, prove that $\mathcal{V}_j = \mathcal{V}_J \oplus \bigoplus_{k=0}^{J-j-1} \mathcal{W}_{J-k}$. You must explain all the steps throughout the proof carefully. (15 pts.)

1)
$$s(t) = c_0^{(0)}\phi(t) + d_0^{(0)}\psi(t) + d_0^{(-1)}\psi(2t) + d_1^{(-1)}\psi(2t-1)$$

$$c_0^{(0)} = \langle t^3, \phi(t)\rangle = \int_0^1 t^3\,dt = \frac{1}{4}$$

$$d_0^{(0)} = \langle t^3, \psi(t)\rangle = \int_0^{1/2} t^3\,dt - \int_{1/2}^1 t^3\,dt = \frac{-7}{32}$$

$$d_0^{(-1)} = \langle t^3, \psi(2t)\rangle = \int_0^{1/4} t^3\,dt - \int_{1/4}^{1/2} t^3\,dt = -\frac{7}{512}$$

$$d_1^{(-1)} = \langle t^3, \psi(2t-1)\rangle = \int_{1/2}^{3/4} t^3\,dt - \int_{3/4}^{1} t^3\,dt = \frac{-55}{512}$$

2)
$$v_{j-1} = v_j \oplus w_j$$ (direct sum prop)

$$v_j = V_J \oplus W_J \oplus W_{J-1} \oplus \cdots \oplus W_{j+1}$$ (repeated application by decomposition)

$$v_j = V_J \oplus \bigoplus_{k=0}^{J-j-1} W_{J-k}$$

PROBLEM 3: Suppose you are given a color image of size $N \times N$ specified in terms of R, G and B attributes. Note that R, G, B are red, green and blue colors used for describing a pixel within the image as in Figure 1. You can conveniently ignore the intensity attribute throughout this problem. Devise an algorithm for obtaining a *monochromatic* i.e., single color image from the color image using an appropriate linear transformation method. You need to describe the algorithmic steps clearly i.e., specifying the inputs, the procedure and outputs. You must indicate the dimensions of all the vectors and matrices while describing your procedure.

NOTE: You can make reasonable assumptions on any of the variables needed for the algorithm, but must clearly state them. (25 pts.)

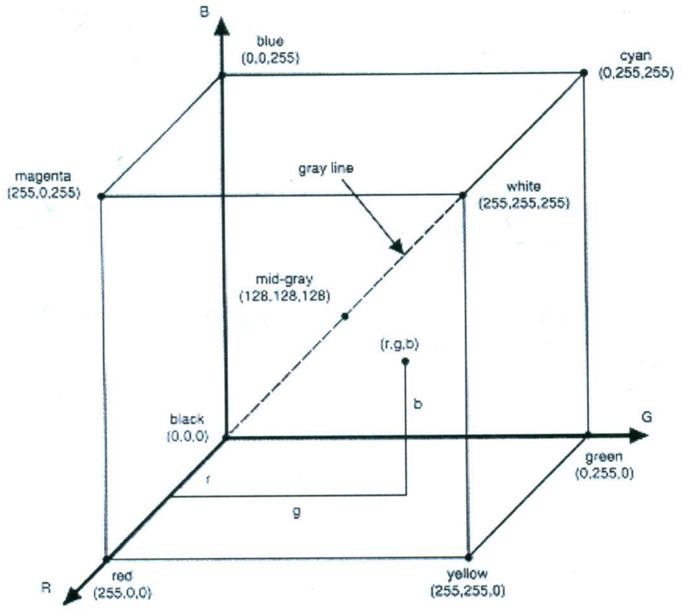

FIGURE 1. Color representation using R, G, B basis.

Inputs: $I_{N \times N}$, totally N^2 pixels.
Each pixel $I : [p_{ij}]$ is a vector with 3 coordinates.

Step 1: Form $A = \begin{bmatrix} v_1 \\ \vdots \\ v_{N \times N} \end{bmatrix}_{N^2 \times 3}$

Step 2: Calculate $\mu(A)_{1\times 3}$ and $\mathrm{cov}(A)_{3\times 3}$

Step 3: Sort the eigen values and eigen vectors corresponding to cov(A).

Step 4: Project all points $\bar{v}_i$ in the direction of the eigen vectors. Corresponding to $\bar{r}, \bar{g}$ & $\bar{b}$
$v_i' = \langle v_i, \hat{e} \rangle \hat{e}$. $_{1\times 3}$

Step 5: Stack all the projected points in a raster way (inverse)

$$\hat{I} = \text{inv raster} \begin{bmatrix} v_1' \\ \vdots \\ v_{N\times N}' \end{bmatrix}_{N^2 \times 3}$$

$$\hat{I} = \begin{bmatrix} v_{1,1}' & \cdots & v_{1,N}' \\ & & \\ v_{N,1}' & & v_{N,N}' \end{bmatrix}_{N \times N}$$

(monochromatic image

Each $v_{i,j}'$ in $\hat{I}$ is a 1×3 vector.

PROBLEM 4: You are given a Padé model having 3 poles as $H_1(z) = \frac{1}{1+2z^{-1}+z^{-2}+3z^{-3}}$.

(a) How many input samples does this model exactly match? (2 pts.)

(b) Determine the input samples that match with the model exactly as in the previous part. (15 pts.)

(c) What would be the input samples if the Padé model had 3 zeros as $H_2(z) = 1 + 2z^{-1} + z^{-2} + 3z^{-3}$? (5 pts.)

(d) Suppose ϵ_1 and ϵ_2 are the modeling errors for $H_1(z)$ and $H_2(z)$ for some infinitely long unknown input whose first few samples were to match. Justify if the claim $\epsilon_1 \geq \epsilon_2$ is true/false. A countercase is enough to make it false. (3 pts.)

a) We need 4 samples to exactly match

$p = 3 \quad q = 0 \quad$ i.e., $p + q + 1$.

b) $h(n) + \sum_{k=1}^{p} a_p(k)\, h(n-k) = b(0)$

$x(n) = h(n)$ for Pade till $p+q$ from 0.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x(0) & & \emptyset & \\ x(1) & x(0) & & \\ x(2) & x(1) & x(0) & \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

Solving $x(0) = 1 \quad x(1) = -2$

$x(2) = 3 \quad x(3) = -7$

c) Samples are just $\{ \underset{\uparrow_0}{1} \;\; 2 \;\; 1 \;\; 3 \}$

d) <u>No</u> ϵ_1 need not be $\geq \epsilon_2$

In one case data outside of 4 samples for $H_2(z)$ can be non zero. These samples under $H_1(z)$ can be yielding less error than $H_2(z)$.