

INDIAN INSTITUTE OF SCIENCE
E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING
HOME WORK #3 - SOLUTIONS, FALL 2014

INSTRUCTOR: SHAYAN G. SRINIVASA
 TEACHING ASSISTANTS: ANKUR RAINA, CHAITANYA KUMAR MATCHA

Problem 1. (5.16) We use following properties of determinant under elementary row operations:

- (a) When a row of a matrix is scaled by a scalar factor, its determinant is also scaled by the same factor.
- (b) When two rows of a matrix are exchanged, the sign of its determinant changes.

By scaling the last r rows of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ z^{-1}\mathbf{I}_r & \mathbf{0} \end{bmatrix}$$

by the factor z , we have

$$\begin{aligned} z^r \det(\mathbf{A}) &= \det\left(\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ \mathbf{I}_r & \mathbf{0} \end{bmatrix}\right) \\ \implies \det(\mathbf{A}) &= z^{-r} \det\left(\begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ \mathbf{I}_r & \mathbf{0} \end{bmatrix}\right). \end{aligned}$$

Note that the the matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-r} \\ \mathbf{I}_r & \mathbf{0} \end{bmatrix}$$

can be transformed to \mathbf{I}_M by using the row exchange operations. Therefore $\det(\mathbf{B})$ is of the form $\pm \det(\mathbf{I}_M) = \pm 1$.

Hence, $\det(\mathbf{A})$ is of the form $\pm z^{-r}$.

Remark:

The matrix \mathbf{B} can be transformed to \mathbf{I}_M using $r(M-r)$ row exchange operations using following procedure (bubble sort algorithm). For $i = 1$ to r , shift the $(M-r+i)^{th}$ row to the top by $(M-r)$ rows bringing down the rows $[\mathbf{0} \ \mathbf{I}_{M-r}]$ by 1 position. This involves $(M-r)$ row exchanges as given by $\mathbf{R}_{M-r+i-j+1} \leftrightarrow \mathbf{R}_{M-r+i-j}$ for $j = 1$ to $(M-r)$. Note that $r(M-r)$ is not the least number of row exchange operations.

Therefore, $\det(\mathbf{A}) = (-1)^{r(M-r)} z^{-r}$.

Problem 2. (5.20) Define

$$\begin{aligned} \mathbf{E}(z) &= [E_0(z) \ E_1(z) \ \cdots \ E_{M-1}(z)]^T, \\ \mathbf{H}(z) &= [H_0(z) \ H_1(z) \ \cdots \ H_{M-1}(z)]^T, \\ \mathbf{\Lambda}(z) &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z^{-1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & z^{-(M-1)} \end{bmatrix}. \end{aligned}$$

We have

$$\mathbf{H}(z) = \mathbf{W}\mathbf{\Lambda}(z)\mathbf{E}(z^M).$$

(Part a)

$$\sum_{k=0}^{M-1} |H_k(e^{j\omega})|^2 = (\mathbf{H}(e^{j\omega}))^\dagger \mathbf{H}(e^{j\omega})$$

$$\begin{aligned}
&= (\mathbf{E}(e^{jM\omega}))^\dagger (\mathbf{\Lambda}(e^{j\omega}))^\dagger \mathbf{W}^\dagger \mathbf{W} \mathbf{\Lambda}(e^{j\omega}) \mathbf{E}(e^{jM\omega}) \\
&= M (\mathbf{E}(e^{jM\omega}))^\dagger (\mathbf{\Lambda}(e^{j\omega}))^\dagger \mathbf{\Lambda}(e^{j\omega}) \mathbf{E}(e^{jM\omega}) \quad (\mathbf{W}^\dagger \mathbf{W} = M\mathbf{I}) \\
&= M (\mathbf{E}(e^{jM\omega}))^\dagger \mathbf{E}(e^{jM\omega}) \quad \left((\mathbf{\Lambda}(e^{j\omega}))^\dagger \mathbf{\Lambda}(e^{j\omega}) = \mathbf{I} \right) \\
&= M \sum_{k=0}^{M-1} |E_k(e^{jM\omega})|^2 \\
&= M \sum_{k=0}^{M-1} 1 \quad (|E_k(z^{j\omega})| = 1 \forall \omega) \\
&= M^2.
\end{aligned}$$

Therefore, the analysis filters $H_k(z), k = 0, 1, \dots, (M-1)$ are power complementary.

(Part b) If $W = e^{-j\frac{2\pi}{M}}$, then

$$H_k(z) = z^{-k} \sum_{i=0}^{M-1} z^{-i} W^{ik} E_i(z^M), \quad k = 1, 2, \dots, M-1.$$

Therefore,

$$\begin{aligned}
H_k(z) \tilde{H}_k(z) &= (z^{-k} \mathbf{w}_k^T \mathbf{E}(z^M)) \left(z^k (\tilde{\mathbf{E}}(z^M))^T \mathbf{w}_k^* \right) \\
&= \mathbf{w}_k^T (\mathbf{E}(z^M) \mathbf{E}^\dagger(z^M)) \mathbf{w}_k^*. \\
\Rightarrow \sum_{i=0}^{M-1} \tilde{H}_k(zW^i) H_k(zW^i) &= \sum_{i=0}^{M-1} \left(z^i \sum_{p=0}^{M-1} (zW^i)^p W^{-pk} \tilde{E}_p(z^M W^{iM}) \right) \left(z^{-i} \sum_{q=0}^{M-1} (zW^i)^{-q} W^{qk} E_q(z^M W^{iM}) \right) \\
&= \sum_{i=0}^{M-1} \left(\sum_{p=0}^{M-1} z^p W^{-p(k-i)} \tilde{E}_p(z^M) \right) \left(\sum_{q=0}^{M-1} z^{-q} W^{q(k+i)} E_q(z^M) \right) \\
&= \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} \sum_{i=0}^{M-1} z^{p-q} W^{(q-p)k} W^{(p-q)i} \tilde{E}_p(z^M) E_q(z^M) \\
&= \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} \left(z^{p-q} W^{(q-p)k} \tilde{E}_p(z^M) E_q(z^M) \sum_{i=0}^{M-1} W^{(p-q)i} \right) \quad (\text{Rearranging summations}) \\
&= \sum_{p=0}^{M-1} \sum_{q=0}^{M-1} \left(z^{p-q} W^{(q-p)k} \tilde{E}_p(z^M) E_q(z^M) M\delta(p-q) \right) \quad \left(\text{Using } \sum_{i=0}^{M-1} W^{ik} = M\delta(k) \right) \\
&= M \sum_{p=0}^{M-1} \tilde{E}_p(z^M) E_p(z^M) \quad \left(\text{Using } \delta(p-q) = \begin{cases} 1, & p=q, \\ 0, & p \neq q. \end{cases} \right) \\
&= M \sum_{p=0}^{M-1} 1 \quad \left(\text{On unit circle, } \tilde{E}_p(z^M) E_p(z^M) = |E_p(z^M)| = 1. \right) \\
\Rightarrow \sum_{i=0}^{M-1} \tilde{H}_k(zW^i) H_k(zW^i) &= M^2.
\end{aligned}$$

Therefore, $H_k(z) \tilde{H}_k(z)$ satisfies M^{th} band property. Hence, each analysis filter is a spectral factor of an M^{th} band filter.

(Part c) The synthesis bank structure is shown in the Figure 1.

We use the following results proved in the proposition 1 at the end of this document.

(a) The system is alias free iff $E_k(z) R_k(z) = S(z), k = 0, 1, \dots, M-1$ for some $S(z) \neq 0$.

(b) The overall system transfer function under this condition is $T(z) = z^{-(M-1)} S(z^M)$.

Since we need a stable synthesis bank and no information on the zeros of $E_k(z)$ are provided, we require $E_k(z)$

to be a factor of $S(z) \forall k$. Therefore, we choose $S(z) = \prod_{k=0}^{M-1} E_k(z)$. With this choice, the synthesis bank filters are

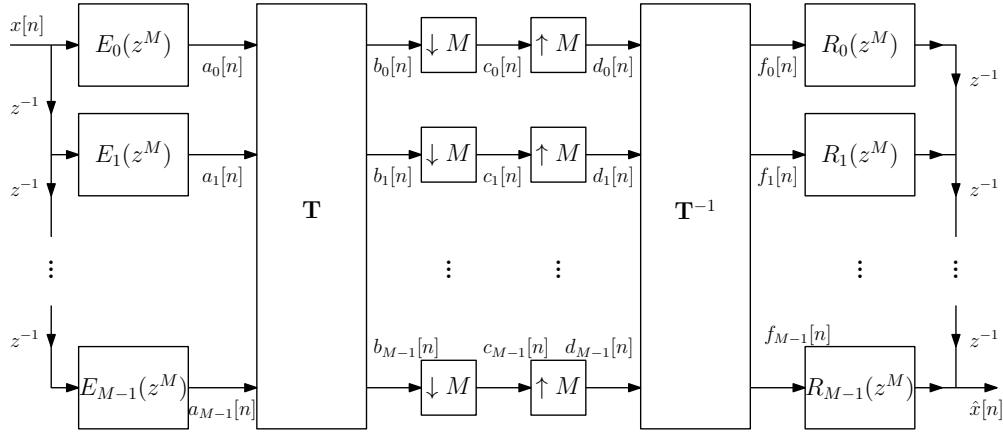


FIGURE 1. A generic analysis and synthesis filter bank system. Here $\mathbf{T} = \mathbf{W}^*$, where \mathbf{W} is the $M \times M$ DFT matrix. The system is alias free if $E_k(z) R_k(z) = S(k)$, $k = 0, 1, \dots, M-1$, giving the overall system transfer function as $T(z) = z^{-(M-1)} S(z^M)$.

given by

$$R_k(z) = \prod_{i \neq k} E_i(z), \quad k = 0, 1, \dots, M-1.$$

The overall system response is

$$T(z) = z^{-(M-1)} \prod_{k=0}^{M-1} E_k(z^M).$$

The amplitude response is

$$\begin{aligned} |T(e^{j\omega})| &= \left| e^{-j\omega(M-1)} \prod_{k=0}^{M-1} E_k(e^{-j\omega M}) \right| \\ &= \left| e^{-j\omega(M-1)} \right| \prod_{k=0}^{M-1} |E_k(e^{-j\omega M})| \\ |T(e^{j\omega})| &= 1. \end{aligned}$$

Hence, this choice of synthesis filter bank gives alias-free output without amplitude distortion.

Problem 3. (5.13)

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{\infty} s_a(nT_1) \delta(t - nT_1) \\ &= \sum_{n=-\infty}^{\infty} s_a(t) \delta(t - nT_1) \quad \left(\text{Using } \delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \right) \\ &= s_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_1) \\ &= s_a(t) \sum_{n=-\infty}^{\infty} e^{-j2\pi \frac{nt}{T_1}} \quad \left(\text{Using } \sum_{n=-\infty}^{\infty} \delta(t - nT_1) = \sum_{n=-\infty}^{\infty} e^{-j2\pi \frac{nt}{T_1}} \right) \\ &= \sum_{n=-\infty}^{\infty} s_a(t) e^{-j2\pi \frac{nt}{T_1}} \\ \Rightarrow S(j\Omega) &= \int_{-\infty}^{\infty} s_a(t) e^{j\Omega t} dt \end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \left(e^{j\Omega t} \sum_{n=-\infty}^{\infty} s_a(t) e^{-j2\pi \frac{nt}{T_1}} \right) \\
&= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} s_a(t) e^{j\left(\Omega - 2\pi \frac{n}{T_1} t\right)} \right) \quad (\text{Assume the exchange is possible.}) \\
S(j\Omega) &= \sum_{n=-\infty}^{\infty} S_a \left(j\Omega - j \frac{2\pi n}{T_1} \right).
\end{aligned}$$

(Part a) We have

$$S_{a,k}(j\Omega) = X_a(j\Omega) H_{a,k}(j\Omega), \quad k = 0, 1.$$

$$\begin{aligned}
\Rightarrow S_k(j\Omega) &= \sum_{n=-\infty}^{\infty} S_{a,k} \left(j\Omega - j \frac{2\pi n}{T_1} \right) \\
S_k(j\Omega) &= \sum_{n=-\infty}^{\infty} X_a \left(j\Omega - j \frac{2\pi n}{T_1} \right) H_{a,k} \left(j\Omega - j \frac{2\pi n}{T_1} \right), \quad k = 0, 1.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \hat{X}(j\Omega) &= F_{a,0}(j\Omega) S_0(j\Omega) + F_{a,1}(j\Omega) S_1(j\Omega) \\
&= \sum_{n=-\infty}^{\infty} X_a \left(j\Omega - j \frac{2\pi n}{T_1} \right) \left(F_{a,0}(j\Omega) H_{a,0} \left(j\Omega - j \frac{2\pi n}{T_1} \right) + F_{a,1}(j\Omega) H_{a,1} \left(j\Omega - j \frac{2\pi n}{T_1} \right) \right).
\end{aligned}$$

(Part b) Given $T_1 = 2T$, $T = \frac{2\pi}{\Theta}$, $\Theta = 2\sigma$. Substituting $T_1 = \frac{2\pi}{\sigma}$, we have

$$\hat{X}(j\Omega) = \sum_{n=-\infty}^{\infty} X_a(j(\Omega - n\sigma)) (F_{a,0}(j\Omega) H_{a,0}(j(\Omega - n\sigma)) + F_{a,1}(j\Omega) H_{a,1}(j(\Omega - n\sigma))). \quad (1)$$

Given that $X(j\Omega) = 0$ for $|\Omega| \geq \sigma$. Therefore, $X(j(\Omega - n\sigma)) \neq 0$ only if

$$\begin{aligned}
-\sigma &< \Omega - n\sigma < \sigma \\
\Rightarrow \frac{\Omega}{\sigma} - 1 &< n < \frac{\Omega}{\sigma} + 1
\end{aligned}$$

i.e., n can be an integer in an open interval of length 2. Hence, n can take at most two values. Therefore, only 2 terms can be non-zero for a given value of Ω .

(Part c) Given that $|F_{a,k}(j\Omega)| = 0$ for $|\Omega| \geq \sigma$. From (1), $\hat{X}(z) = 0$ for $|\Omega| \geq \sigma$.

Case $-\sigma < \Omega \leq 0$:

$n \in \left(\frac{\Omega}{\sigma} - 1, \frac{\Omega}{\sigma} + 1\right) \Rightarrow n \in (-2, 1) \Rightarrow n \in \{-1, 0\}$. In this case,

$$\begin{aligned}
\hat{X}(j\Omega) &= X_a(j\Omega) (F_{a,0}(j\Omega) H_{a,0}(j\Omega) + F_{a,1}(j\Omega) H_{a,1}(j\Omega)) \\
&\quad + X_a(j\Omega + j\sigma) (F_{a,0}(j\Omega) H_{a,0}(j\Omega + j\sigma) + F_{a,1}(j\Omega) H_{a,1}(j\Omega + j\sigma)) \\
\hat{X}(j\Omega) &= \begin{bmatrix} X_a(j\Omega) & X_a(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega + j\sigma) & H_{a,1}(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix}.
\end{aligned}$$

Therefore, the output is free from aliasing and distortion if

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega + j\sigma) & H_{a,1}(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}. \quad (2)$$

Case $0 < \Omega < \sigma$:

$n \in \left(\frac{\Omega}{\sigma} - 1, \frac{\Omega}{\sigma} + 1\right) \Rightarrow n \in (-1, 2) \Rightarrow n \in \{0, 1\}$. In this case,

$$\begin{aligned}
\hat{X}(j\Omega) &= X_a(j\Omega) (F_{a,0}(j\Omega) H_{a,0}(j\Omega) + F_{a,1}(j\Omega) H_{a,1}(j\Omega)) \\
&\quad + X_a(j\Omega - j\sigma) (F_{a,0}(j\Omega) H_{a,0}(j\Omega - j\sigma) + F_{a,1}(j\Omega) H_{a,1}(j\Omega - j\sigma)) \\
\hat{X}(j\Omega) &= \begin{bmatrix} X_a(j\Omega) & X_a(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega - j\sigma) & H_{a,1}(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix}.
\end{aligned}$$

Therefore, the output is free from aliasing and distortion if

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega - j\sigma) & H_{a,1}(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}. \quad (3)$$

(Part d) $H_{a,0}(j\Omega) = 1$ and $H_{a,1}(j\Omega) = j\Omega T_1$. Using $\sigma T_1 = 2\pi$, In this case the matrices in (2) and (3) are

$$\begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 + j2\pi \end{bmatrix} \text{ and } \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 - j2\pi \end{bmatrix}.$$

Their determinants are $j2\pi$ and $-j2\pi$ respectively. Therefore, the matrices are non-singular.

Case $-\sigma < \Omega \leq 0$:

$$\begin{aligned} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 + j2\pi \end{bmatrix}^{-1} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\ &= \frac{1}{j2\pi} \begin{bmatrix} j(\Omega T_1 + 2\pi) & -j\Omega T_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} \frac{(\Omega T_1 + 2\pi)T_1}{2\pi} \\ j\frac{T_1}{2\pi} \end{bmatrix} \end{aligned}$$

Case $0 \leq \Omega < \sigma$:

$$\begin{aligned} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 - j2\pi \end{bmatrix}^{-1} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\ &= \frac{-1}{j2\pi} \begin{bmatrix} j(\Omega T_1 - 2\pi) & -j\Omega T_1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} \frac{(2\pi - \Omega T_1)T_1}{2\pi} \\ -j\frac{T_1}{2\pi} \end{bmatrix}. \end{aligned}$$

Therefore, the synthesis filters are given by

$$F_{a,0}(j\Omega) = \begin{cases} \frac{2\pi}{\sigma} \left(1 - \frac{1}{\sigma} |\Omega|\right), & |\Omega| < \sigma \\ 0 & |\Omega| \geq \sigma \end{cases}$$

$$F_{a,1}(j\Omega) = \begin{cases} -j\frac{2}{\sigma} \text{sign}(\Omega), & |\Omega| < \sigma \\ 0 & |\Omega| \geq \sigma \end{cases}$$

Define ideal low-pass filter and the corresponding time-domain sinc(\cdot) function as

$$W(j\Omega) = \begin{cases} 1 & |\Omega| < \frac{\sigma}{2} \\ 0 & |\Omega| \geq \frac{\sigma}{2} \end{cases},$$

$$w(t) = \frac{\sigma \sin(\sigma t/2)}{2\pi (\sigma t/2)}.$$

We can write $F_{a,0}(j\Omega) = \frac{2\pi}{\sigma^2} W(j\Omega) * W(j\Omega)$. Therefore,

$$\begin{aligned} f_{a,0}(t) &= \frac{2\pi}{\sigma^2} \times 2\pi (w_1(t))^2 \quad (\mathcal{F}^{-1}(X_1(j\Omega) * X_1(j\Omega)) = 2\pi x_1(t) x_2(t)) \\ \implies f_{a,0}(t) &= 4 \sin^2(\sigma t/2) / \sigma^2 t^2. \end{aligned}$$

Similarly, $F_{a,1}(j\Omega) = -j\frac{2}{\sigma} (-W(j(\Omega + \frac{\sigma}{2})) + W(j(\Omega - \frac{\sigma}{2})))$. Therefore,

$$\begin{aligned} f_{a,1}(t) &= -j\frac{2}{\sigma} \left(-w(t) e^{-j\frac{\sigma t}{2}} + w(t) e^{j\frac{\sigma t}{2}} \right) \\ &= \frac{4}{\sigma} w(t) \sin(\sigma t/2) \\ f_{a,1}(t) &= \frac{1}{2\pi} (4 \sin^2(\sigma t/2) / \sigma^2 t). \end{aligned}$$

Proposition 1. For the analysis and synthesis filter bank system in the Figure 1,

- The overall system transfer function is independent of the non-singular matrix \mathbf{T} .
- The output of the system is alias-free iff $E_k(z) R_k(z) = S(z)$, $k = 0, 1, \dots, M-1$ for some $S(z) \neq 0$.
- Under the alias-free condition, the overall system transfer function is $T(z) = z^{-(M-1)} S(z^M)$.

Proof. Let the intermediate signals in the system be as indicated in the Figure 1. Define

$$\begin{aligned}
\mathbf{A}(z) &= [A_0(z) \ A_1(z) \ \cdots \ A_{M-1}(z)], \\
\mathbf{B}(z) &= [B_0(z) \ B_1(z) \ \cdots \ B_{M-1}(z)], \\
\mathbf{C}(z) &= [C_0(z) \ C_1(z) \ \cdots \ C_{M-1}(z)], \\
\mathbf{D}(z) &= [D_0(z) \ D_1(z) \ \cdots \ D_{M-1}(z)], \\
\mathbf{F}(z) &= [F_0(z) \ F_1(z) \ \cdots \ F_{M-1}(z)], \\
\mathbf{E}(z) &= [E_0(z) \ E_1(z) \ \cdots \ E_{M-1}(z)], \\
\mathbf{R}(z) &= [A_0(z) \ A_1(z) \ \cdots \ A_{M-1}(z)] \\
\text{and } \mathbf{\Lambda}(z) &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & z^{-1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & z^{-(M-1)} \end{bmatrix}.
\end{aligned}$$

Using this notation, we derive the output $\hat{X}(z)$ as the function of input $X(z)$ as follows:

$$\mathbf{A}(z) = X(z)\mathbf{\Lambda}(z)\mathbf{E}(z^M). \quad (4)$$

$$\mathbf{B}(z) = \mathbf{TA}(z)$$

$$\mathbf{B}(z) = X(z)\mathbf{TA}(z)\mathbf{E}(z^M). \quad (5)$$

$$\begin{aligned}
\mathbf{C}(z) &= \frac{1}{M} \sum_{i=0}^{M-1} \mathbf{B}\left(z^{\frac{1}{M}}W^i\right) \\
&= \frac{1}{M} \sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}}W^i\right) \mathbf{TA}\left(z^{\frac{1}{M}}W^i\right) \mathbf{E}(zW^{iM}) \\
&= \frac{1}{M} \sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}}W^i\right) \mathbf{TA}\left(z^{\frac{1}{M}}W^i\right) \mathbf{E}(z) \\
\mathbf{C}(z) &= \frac{1}{M} \mathbf{T} \left(\sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}}W^i\right) \mathbf{\Lambda}\left(z^{\frac{1}{M}}W^i\right) \right) \mathbf{E}(z). \quad (6)
\end{aligned}$$

$$\mathbf{D}(z) = \mathbf{C}(z^M)$$

$$\mathbf{D}(z) = \frac{1}{M} \mathbf{T} \left(\sum_{i=0}^{M-1} X(zW^i) \mathbf{\Lambda}(zW^i) \right) \mathbf{E}(z^M). \quad (7)$$

$$\mathbf{F}(z) = \mathbf{T}^{-1}\mathbf{D}(z)$$

$$\mathbf{F}(z) = \frac{1}{M} \mathbf{T} \left(\sum_{i=0}^{M-1} X(zW^i) \mathbf{\Lambda}(zW^i) \right) \mathbf{E}(z^M). \quad (8)$$

We can write,

$$[z^{-(M-1)}R_0(z^M) \ z^{-(M-2)}R_1(z^M) \ \cdots \ R_1(z^M)] = (\mathbf{R}(z^M))^T (z^{M-1}\mathbf{\Lambda}(z))^{-1}.$$

Using this,

$$\begin{aligned}
\hat{X}(z) &= (\mathbf{R}(z^M))^T z^{-(M-1)} (\mathbf{\Lambda}(z))^{-1} \mathbf{F}(z) \\
&= \frac{z^{-(M-1)}}{M} (\mathbf{R}(z^M))^T \left(\sum_{i=0}^{M-1} X(zW^i) (\mathbf{\Lambda}(z))^{-1} \mathbf{\Lambda}(zW^i) \right) \mathbf{E}(z^M) \\
&= \frac{z^{-(M-1)}}{M} (\mathbf{R}(z^M))^T \left(\sum_{i=0}^{M-1} X(zW^i) \mathbf{\Lambda}(W^i) \right) \mathbf{E}(z^M)
\end{aligned}$$

$$\begin{aligned}
&= \frac{z^{-(M-1)}}{M} \sum_{p=0}^{M-1} \sum_{i=0}^{M-1} R_p(z) X(zW^i) W^{-ip} E_p(z^M) \\
\hat{X}(z) &= \frac{z^{-(M-1)}}{M} \sum_{i=0}^{M-1} \left(X(zW^i) \sum_{p=0}^{M-1} R_p(z^M) E_p(z^M) W^{-ip} \right). \tag{9}
\end{aligned}$$

In matrix representation, the output can be written as

$$\hat{X}(z) = \frac{z^{-(M-1)}}{M} [X(z) \quad X(zW) \quad \dots \quad X(W^{M-1})] \mathbf{W} \begin{bmatrix} E_0(z^M) R_0(z^M) \\ E_1(z^M) R_1(z^M) \\ \vdots \\ E_{M-1}(z^M) R_{M-1}(z^M) \end{bmatrix}. \tag{10}$$

To obtain alias-free output, the necessary condition is

$$\frac{1}{M} \mathbf{W} \begin{bmatrix} E_0(z^M) R_0(z^M) \\ E_1(z^M) R_1(z^M) \\ \vdots \\ E_{M-1}(z^M) R_{M-1}(z^M) \end{bmatrix} = \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

for some $S(z) \neq 0$. Since $\mathbf{W}^* \mathbf{W} = M\mathbf{I}$, we have

$$\begin{bmatrix} E_0(z^M) R_0(z^M) \\ E_1(z^M) R_1(z^M) \\ \vdots \\ E_{M-1}(z^M) R_{M-1}(z^M) \end{bmatrix} = \mathbf{W}^* \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} S(z^M) \\ S(z^M) \\ \vdots \\ S(z^M) \end{bmatrix} \tag{11}$$

$$\text{i.e., } R_p(z) E_p(z) = S(z), \quad p = 0, 1, \dots, M-1. \tag{12}$$

Under this condition, the output is

$$\begin{aligned}
\hat{X}(z) &= z^{-(M-1)} [X(z) \quad X(zW) \quad \dots \quad X(W^{M-1})] \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\hat{X}(z) &= z^{-(M-1)} S(z^M) X(z). \tag{13}
\end{aligned}$$

Therefore, the transfer function is

$$T(z) = \frac{\hat{X}(z)}{X(z)} = z^{-(M-1)} S(z^M). \tag{14}$$

□