Solutions Key

Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

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Mid Term Exam#1, Fall 2015

Name and SR.No:

Instructions:

- Only four A4 sheets of paper with written notes on both sides are allowed .
- The time duration is 3 hrs.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: Examine if the following statements are true or false with correct reasoning. Random guessing or incorrect reasoning fetches zero credit. A statement is true if it is generic for all cases. A counter example is enough to make it false. All sub-parts of this problem carry equal credit.

(1) Let $\bar{v}$ be the eigenvector for a square matrix A with a corresponding eigenvalue zero. $\bar{v}$ is also an eigenvector for A^n for integers $n > 1$.

(2) Consider the random process $s(t) = A\sin(\omega t)$, where A is Gaussian distributed with mean zero and variance σ^2. The frequency ω is deterministic. $s(t)$ is a stationary process.

(3) Upsampling of a non-zero discrete time signal by a factor $L > 1$ always causes images within the base band.

(4) Consider a discrete time signal $x[n]$ at D samples/s coded at c bits/sample with energy predominantly in the low pass region. Suppose we pass this signal through a two channel QMF bank so that the output of the analysis bank is coded at a bits and b bits per sample in the low and high frequency subbands respectively during transmission. We can achieve a compression in the data rate if $2c > a + b$.

(5) If an LTI system is causal, it is always stable.

(20 pts.)

1) $A\bar{v} = 0 \cdot \bar{v}$ $(\lambda = 0)$ From eigen value eqn

Try $A^2\bar{v} = A \cdot A\bar{v} = A \cdot \vec{0} = 0$

Continuing, $A^n\bar{v} = \bar{0}$ $\Rightarrow$ $\bar{v}$ is also an eigen vector for A^n $n > 1$.

$(\because A^{n-1} A v = 0 \cdot \bar{v})$

$\boxed{\text{TRUE}}$

2) Consider the 3rd moment and higher ... (odd ones $= 0$ / even $\neq 0$)

$E(s^4(t)) = E(A^4)\sin^4(\omega t)$

$E(A^4) \neq 0$ for Gaussian $\Rightarrow$ 4th order statistics is a function of 't'

$\Rightarrow$ Not strict sense stationary.

$\boxed{\text{FALSE}}$

3) DFT is periodic with 2π $\Rightarrow$ $\uparrow$ by L causes those spectra beyond base band to be 'wrapped' in due to compression

$\therefore$ Statement is $\boxed{\text{true}}$

4) Bit rate originally is Dc bits/s

At the subbands, bit rate $= \frac{D}{2}a + \frac{D}{2}b$ due to downsampling by 2

from 2-channel QMF

$\Rightarrow$ Compression rate (CR) $= \dfrac{Dc}{\frac{D}{2}(a+b)}$ $\left(CR > 1 \Rightarrow 2c > a+b\right)$

$\boxed{\text{TRUE}}$

5) Suppose $h[n] = u[n]$ (Causal)

$\sum_{n \geq 0} h(n) = \infty$ (Not stable)

$\boxed{\text{FALSE}}$

PROBLEM 2: This problem has 3 parts.

(1) Let $\{\bar{v}_i\}_{i=1}^n$ be a collection of orthonormal vectors and $\{a_i\}_{i=1}^n$ be scalars. Prove that $||\sum_{i=1}^n a_i\bar{v}_i||^2 = \sum_{i=1}^n |a_i|^2$. (8 pts.)

(2) Let the vector space $\mathcal{V}_1 \perp \mathcal{V}_2$.
(a) Prove that $\mathcal{V}_2$ is a subspace of $\mathcal{V}_1^\perp$. State the converse and verify if it is true.
(b) Prove that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{\bar{0}\}$. (8 pts.)

(3) Consider the signals $f_1(t) = 1$ and $f_2(t) = t$ over $[-1, 1]$. Are they orthonormal? Obtain a closest linear signal approximation to $s(t) = t^2$ over $[-1, 1]$. Plot the approximate signal representation on a signal coordinate system. (9 pts.)

1) Consider $\left(\sum_{i=1}^{n} a_i \bar{v}_i\right)\left(\sum_{i=1}^{n} a_i \bar{v}_i\right)^H$

$= \sum_{i=1}^{n} a_i a_i^* \bar{v}_i \bar{v}_i^H$ $\quad \left(\because \bar{v}_i \bar{v}_j^H = 0, \; \bar{v}_i \bar{v}_i^H = 1\right)$

$= \sum_{i=1}^{n} |a_i|^2$ $\quad \boxed{\Xi}$

(2) (a) Since $\mathcal{V}_1 \perp \mathcal{V}_2 \Rightarrow$ every vector in $\mathcal{V}_2$ is orthogonal to all vectors in $\mathcal{V}_1$ & lies within $\mathcal{V}_1^\perp$ by definition of $\mathcal{V}_1^\perp$.

$\Rightarrow \mathcal{V}_2 \subset \mathcal{V}_1^\perp$

<u>CONVERSE</u> If $\mathcal{V}_2 \subset \mathcal{V}_1^\perp$, then $\mathcal{V}_1 \perp \mathcal{V}_2$

If $\mathcal{V}_2 \subset \mathcal{V}_1^\perp$, every vector in $\mathcal{V}_2$ is in $\mathcal{V}_1^\perp$

$\Rightarrow$ Every vector in $\mathcal{V}_2$ is orthogonal to every vector in $\mathcal{V}_1$

$\Rightarrow \mathcal{V}_1 \perp \mathcal{V}_2$ $\quad \boxed{\Xi}$

(b) Since $\bar{0}$ is part of every vector space, $\mathcal{V}_1 \cap \mathcal{V}_2 = \bar{0}$. But, we must prove that there is no other vector $\bar{v}$ other than $\bar{0}$ in $\mathcal{V}_1 \cap \mathcal{V}_2$. Let us prove by Contradiction..

Suppose $\bar{v} \in \mathcal{V}_1$ & $\bar{v} \in \mathcal{V}_2$ since $\mathcal{V}_1 \perp \mathcal{V}_2$

$\Rightarrow \bar{v} \perp \bar{v} \Rightarrow \langle \bar{v}, \bar{v} \rangle = 0 \Rightarrow \bar{v}$ must be $\bar{0}$ & none else. $\quad \boxed{\Xi}$

3) Consider $f_1(t) = 1 \quad f_2(t) = t$ over $[-1, 1]$

$$\int_{-1}^{1} f_1(t)\, f_2(t)\, dt = 0 \qquad \text{(Orthogonal)}$$

However $\| f_2(t) \| = \left(\int_{-1}^{1} t^2\, dt \right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}} \neq 1$

$\therefore$ They are <u>not</u> <u>orthonormal</u>.

Let us form an orthonormal basis

$$\phi_1(t) = \frac{f_1(t)}{\| f_1(t) \|} = \frac{1}{\sqrt{2}} \qquad -1 \leq t \leq 1$$

$$\phi_2(t) = \frac{f_2(t)}{\| f_2(t) \|} = \frac{f_2(t)}{\left(\sqrt{2/3}\right)} \qquad -1 \leq t \leq 1$$

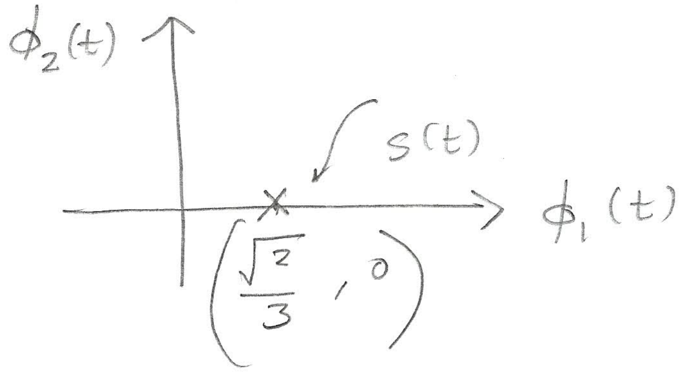
To get the coordinates of $s(t)$ in $\phi_1(t) - \phi_2(t)$ plane, we need

$\langle s(t), \phi_1(t) \rangle$ and $\langle s(t), \phi_2(t) \rangle$

$$\langle s(t), \phi_1(t) \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2\, dt = \frac{\sqrt{2}}{3}$$

$$\langle s(t), \phi_2(t) \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} t^3\, dt = 0$$

(Integral of an odd function)



PROBLEM 3: Consider the structure shown in Figure 1, where $\mathbf{W}$ is the 3×3 DFT matrix. This is a three channel synthesis bank with three filters $F_0(z)$, $F_1(z)$ and $F_2(z)$. (For example, $F_0(z) = Y(z)/Y_0(z)$ with $y_1[n]$ and $y_2[n]$ set to zero.)

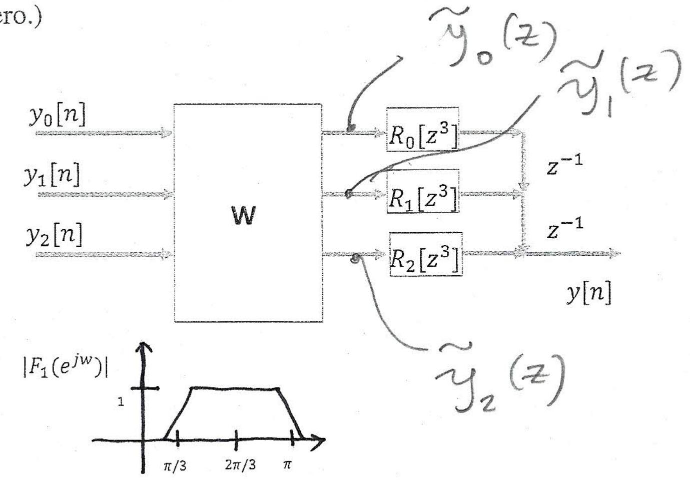


FIGURE 1. Three channel synthesis bank

(1) Assuming $R_0(z) = 1 + z^{-1}, R_1(z) = 1 - z^{-2}, R_2(z) = 2 + 3z^{-1}$, find an expression for the three synthesis filters $F_0(z), F_1(z)$ and $F_2(z)$. (15 pts.)

(2) Let the magnitude of $F_1(z)$ be as shown above. Plot the frequency response of $|F_0(e^{j\omega}|$ and $|F_2(e^{j\omega}|$. (5 pts.)

1) $$Y(z) = \begin{bmatrix} z^{-2}R_0(z^3) & z^{-1}R_1(z^3) & R_2(z^3) \end{bmatrix} \begin{bmatrix} \tilde{y}_0(z) \\ \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix}$$

But $$\begin{bmatrix} \tilde{y}_0(z) \\ \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \begin{bmatrix} y_0(z) \\ y_1(z) \\ y_2(z) \end{bmatrix}$$

where $\omega = e^{-j2\pi/3}$ $\quad \omega^4 = \omega$.

Plug in $y_1(z)$ and $y_2(z) = 0$ to get $F_0(z)$

$$\therefore F_0(z) = z^{-2}R_0(z^3) + z^{-1}R_1(z^3) + R_2(z^3)$$
$$= z^{-2}(1+z^{-3}) + z^{-1}(1-z^{-6}) + 2 + 3z^{-3}$$
$$= 2 + z^{-1} + z^{-2} + 3z^{-3} + z^{-5} - z^{-7}$$

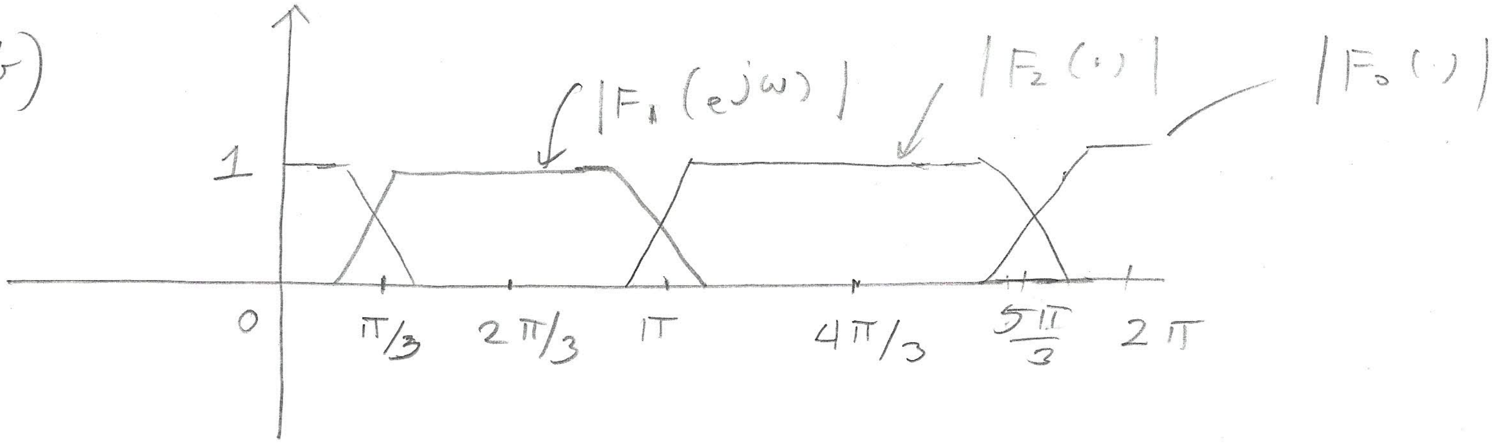
In a similar way setting the other points zero

$$F_1(z) = 2\omega^2 + \omega z^{-1} + z^{-2} + 3\omega^2 z^{-3} + z^{-5} - \omega z^{-7}$$

$$F_2(z) = 2\omega + \omega^2 z^{-1} + z^{-2} + 3\omega z^{-3} + z^{-5} - \omega^2 z^{-7}$$

$$\left.\begin{aligned} F_1(z) &= \omega^2 F_0(\omega z) \\ F_2(z) &= \omega F_0(\omega^2 z) \end{aligned}\right\}$$ These are useful for the next part for spectra sketch.

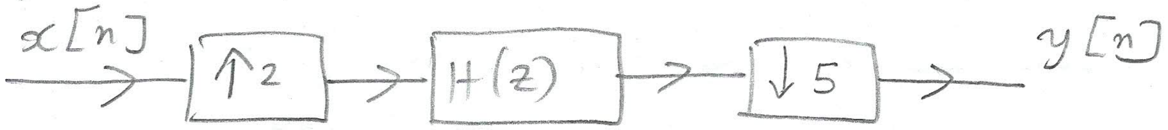
(b)



PROBLEM 4: A certain sampling rate conversion system requires downsampling a signal at 100 Msamples/s to 40 Msamples/s. From first principles, derive a fully efficient architecture using downsamplers and expanders. Sketch the schematic of your multirate system. (25 pts.)

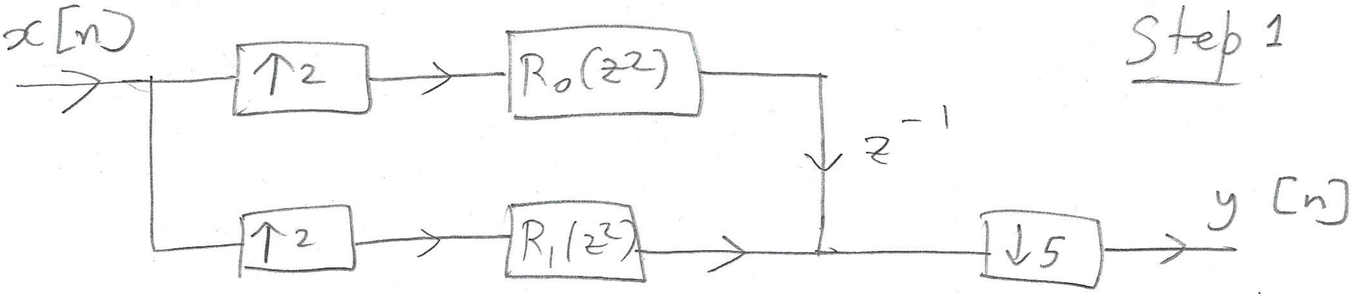
We need a 5 : 2 reduction in rate. $\frac{100}{40} = \frac{5}{2}$

So, the sampling rate conversion system with a filter is

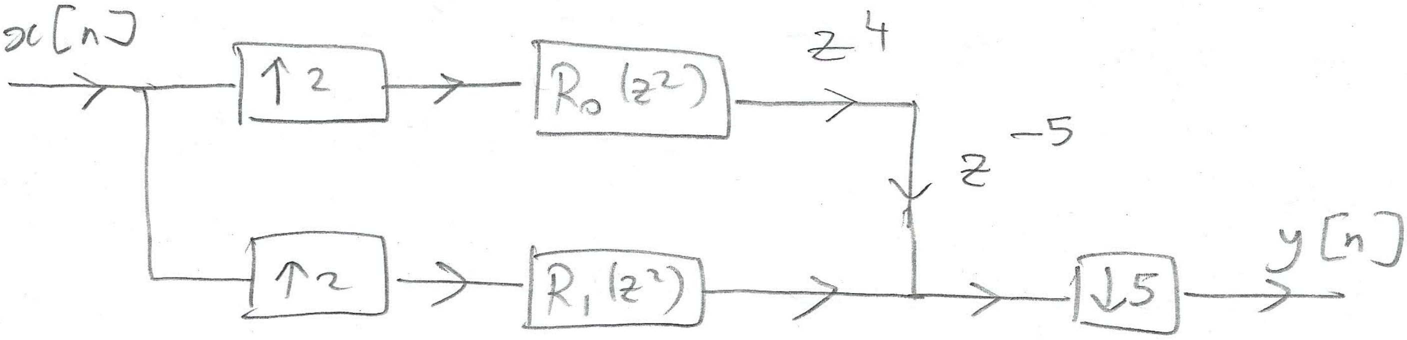


We need to a polyphase architecture similar to Hsiao's work we discussed in class.

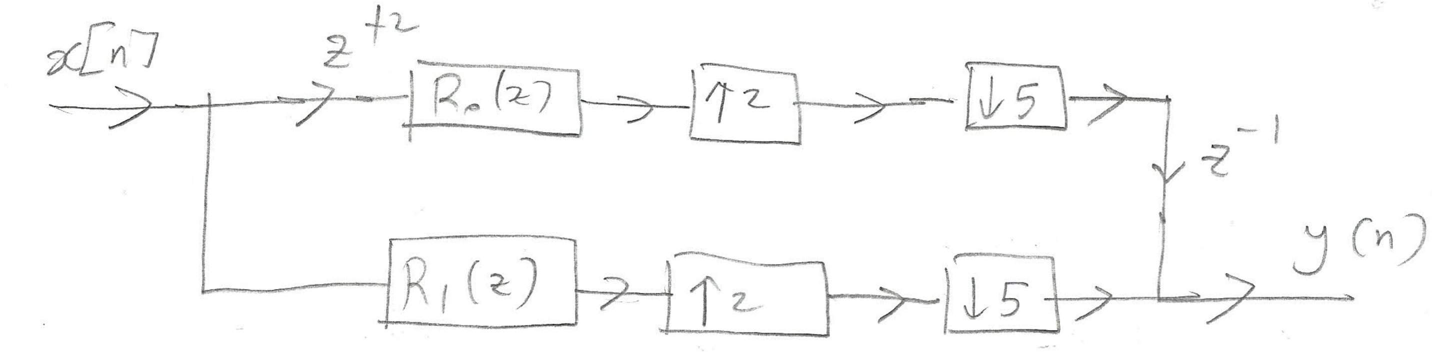
Step 1



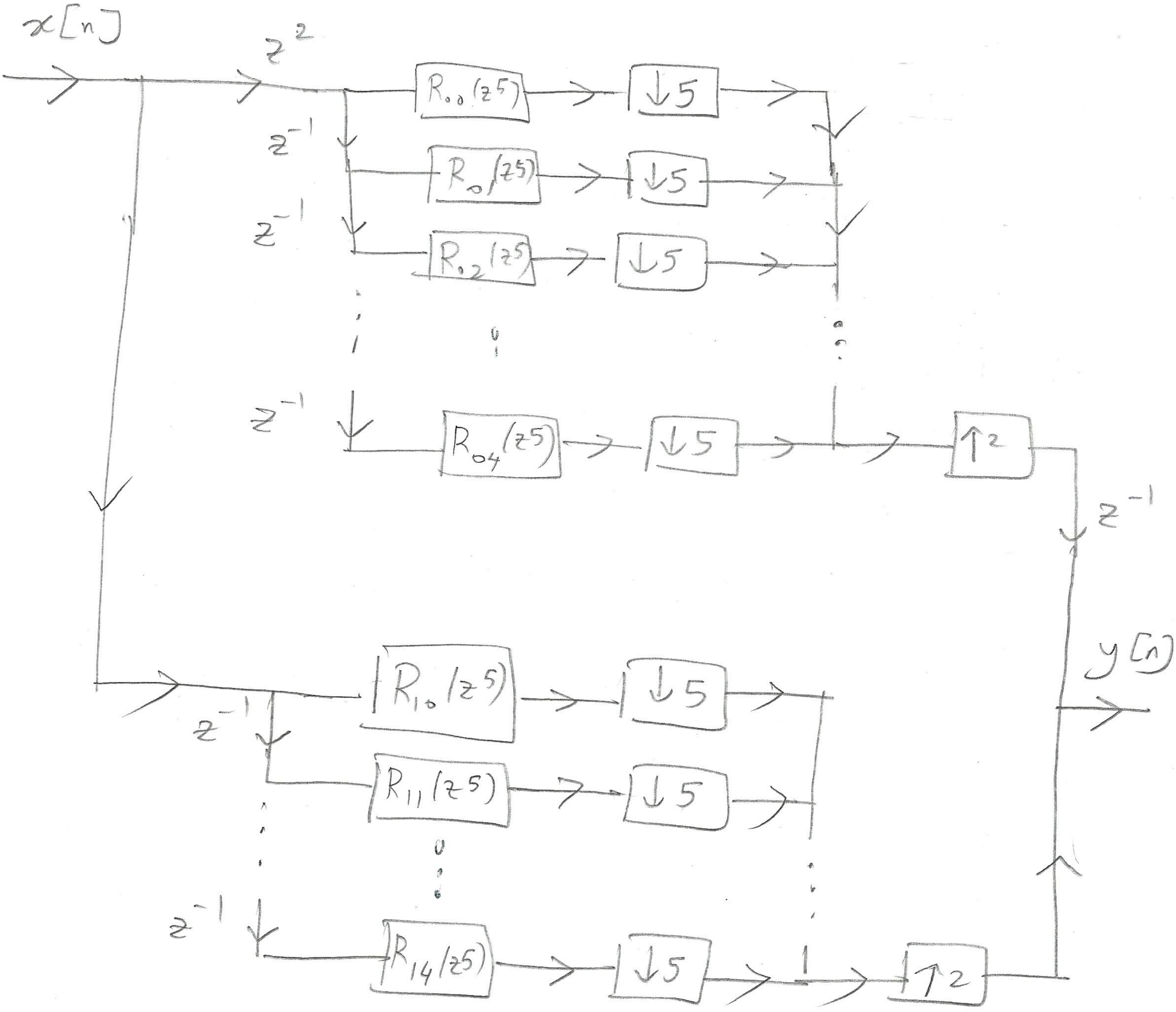
Realize $z^{-1} = z^{-5} \cdot z^{4}$



Now apply noble identities & fold $\boxed{\downarrow 5}$ into the feed forward paths



$\gcd(2, 5) = 1$, We can swap $\uparrow 2$ & $\downarrow 5$



Final architecture

PROBLEM 5: The input to an LTI system is a WSS random process. Is the output WSS? Justify. You can assume that the LTI filter response is absolutely summable. (10 pts.)

$$y(t) = x(t) * h(t) \quad (\because \text{LTI})$$

$$E(y(t)) = E\left(\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right)$$

We can swap $E(\cdot)$ and $\int(\cdot)$ due to absolutely summable & consequence of Tonelli's Conditions.

$$\therefore E(y(t)) = \int_{-\infty}^{\infty} E(x(\tau)\, h(t-\tau))\, d\tau$$

$$= \mu_x \int_{-\infty}^{\infty} h(t-\tau)\, d\tau = \mu_x H(0)$$

dc term of Cont. Fourier tr.

Consider

$E(y(t_1)\, y(t_2))$

From our class notes, this is just.

$$R_{yy}(t_1, t_2) = R_{xx}(t_2 - t_1) * h(t_2 - t_1) * h^*(t_2 - t_1)$$

$\Rightarrow$ It is a WSS process