Solutions Key

Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

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Mid Term Exam#2, Fall 2015

Name and SR.No:

Instructions:

- This is an open book, open notes exam. No wireless allowed.
- The time duration is 3 hrs.
- There are four main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
Total points	

PROBLEM 1: This problem has two parts.

(1) K sensors are placed in a field to take measurements from a source. Each sensor i is sensitive to a frequency band B_i, $1 \leq i \leq K$. The output of each sensor are samples $\{x_i[n]\}_{n=0}^{N-1}$ as shown in Figure 1. Assume that there is correlation between measurements across different sensors due to imperfections even though the frequency bands themselves are non overlapping.
Devise a technique to find the *dominant* frequency band based on the sensor data. Include all the inputs, intermediate variables and outputs in your procedure clearly. Prove the optimality of your solution from first principles. You can make any reasonable assumptions towards a solution on this problem by explicitly stating them. (20 pts.)

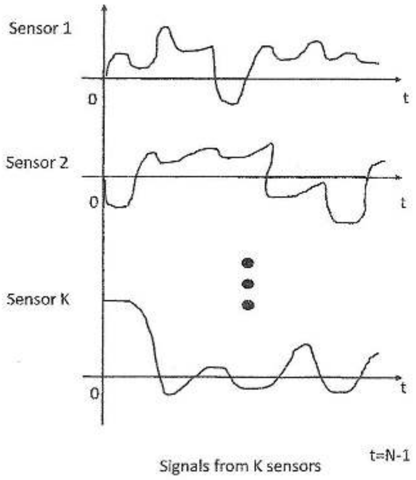


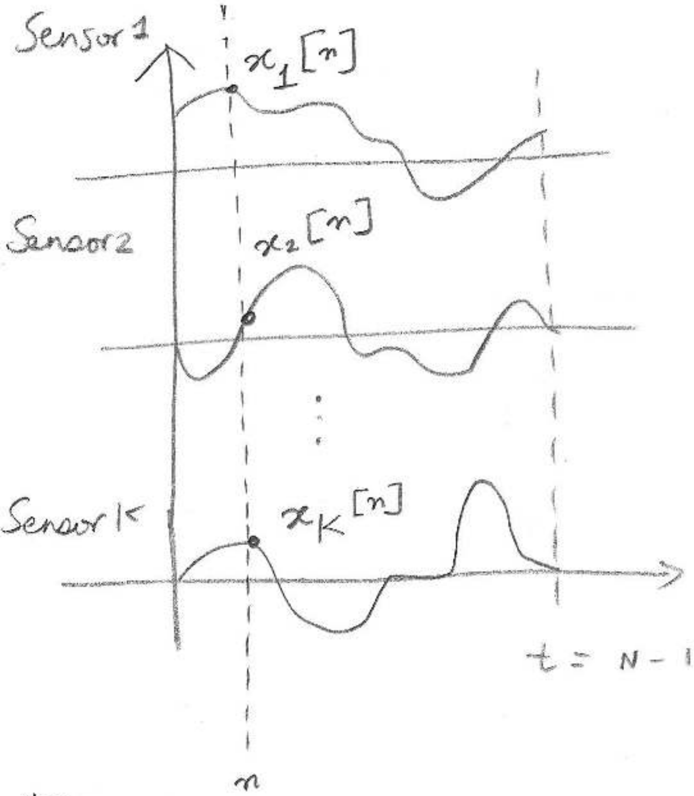
FIGURE 1. Samples from a multi-sensor array.

(2) Consider a 2π periodic piecewise continuous signal $f(t)$ with the existence of k derivatives. Let a_n and b_n denote the Fourier coefficients. Is $|a_n|, |b_n| \leq \frac{1}{\pi n^k} \int_{-\pi}^{\pi} |f^{(k)}(t)|dt$? (5 pts.)

Problem 1 :

(1) We are given K sensors, where each sensor responds to a band B_i, $1 \leq i \leq K$. Each sensor produces samples $\{x_i[n]\}_{n=0}^{N-1}$ that are correlated.

What we need : A procedure to identify the dominant frequency band. You do not have additional information on band details.



Assumptions (Using a linear transform)

1) We are sampling data at the same rate over each sensor o/p consistent to sampling rules.

2) We have no other add'l assumptions on the nature of data from each sensor

3) The data can be noisy, correlated from other sensors... We don't bother since it is handled by the transform.

DATA :

Let us form a vector corresponding to all the sensors $\underline{x}[n] = \left[x_1[n] \;\; x_2[n] \;\; \cdots \;\; x_K[n] \right]^T$. We need to do a PCA on this sampled data and identify the dominant component.

Procedure : KL transform

1) Compute the mean $\mu_{\underline{x}} = E[\underline{x}[n]]_{K \times 1}$

2) Compute the covariance matrix

$$C = E\left[(\underline{x} - \mu_{\underline{x}})(\underline{x} - \mu_{\underline{x}})^T\right]_{K \times K}$$

3) Do an eigen decomposition on C
Let $[A, \Lambda] = \text{eig}(C)$, where Λ is an K eigen val. diagonal matrix corresponding to the different bands and A is the corresponding set of eigen vectors.

<u>NOTE</u>: To get the original data

$$\underline{x} = A^T \underbrace{A(\underline{x} - \mu_{\underline{x}})}_{\text{transformed data after KL. } \underline{y}} + \mu_{\underline{x}}$$

$$\underline{x} = A^T \underline{y} + \mu_{\underline{x}}$$

4) Pick the eigen value that is largest in Λ and the corresponding eigen vector. This <u>should</u> correspond to the dominant channel/band after PCA.

$$\text{band } i = \underset{1 \le i' \le K}{\arg\max} \left(\Lambda(i', i)\right)$$

To prove optimality, we do it straightforwardly from our derivations in class. Since we have adopted a KL procedure here, we need to consider <u>energy</u> in the 1st component/dominant component and show that an eigen filter/vector satisfies the energy maximization under orthonormal constraints.

dominant vector & rest nulled out.

$$\tilde{A} = [\,\underline{a}\ \ 0\ \ 0\ \cdots\,]$$

$$\tilde{A}^H A = \underline{a}^*_{K\times 1}\ \underline{a}^T_{1\times K}$$

$$J = \max_{\{\underline{a}^*\}} E\left(\underline{a}^T \overbrace{(\underline{x}-\mu_{\underline{x}})(\underline{x}-\mu_{\underline{x}})^H}^{\Sigma_x} \underline{a}^*\right) + \lambda\left(1 - \underline{a}^{*T}\underline{a} = 0\right)$$

$$\frac{\partial J}{\partial \underline{a}^*} = 0 \implies \Sigma_x \underline{a} = \lambda \underline{a}$$

<u>Aside</u>:

1) You could have followed any other procedure if you wished to solve this. An alternative way would be to do a wavelet decomposition over each band, compute energy from the transform domain and pick the dominant band. For that you would have to decorrelate and possibly filter out noise as well.

(2) Consider a 2π periodic function which is piecewise continuous

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

$$f^{(k)}(t) = \begin{cases} \left(\sum_{n=1}^{\infty} -a_n n^k \sin(nt) + b_n n^k \cos nt \right) (-1)^{\frac{k-1}{2}} & k \text{ odd} \\ (-1)^{k/2} \sum_{n \geq 1} a_n n^k \cos(nt) + b_n n^k \sin(nt) & k \text{ even} \end{cases}$$

Now
$$|a_n^{(k)}| = \begin{cases} |a_n| n^k & k \text{ even} \\ |b_n| n^k & k \text{ odd} \end{cases}$$

$$|b_n^{(k)}| = \begin{cases} |b_n| n^k & k \text{ even} \\ |a_n| n^k & k \text{ odd} \end{cases}$$

The above are the F.S. Coeffts of $f^{(k)}(t)$

$$|a_n^{(k)}| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} f^{(k)}(t) \cos(nt)\, dt \right|$$

$$\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f^{(k)}(t) \cos(nt)|\, dt$$

$$\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f^{(k)}(t)|\, dt \quad \text{Since } |\cos(nt)| \leq 1$$

Using the relation from $|a_n^{(k)}|$ and $|a_n|$, $|b_n|$

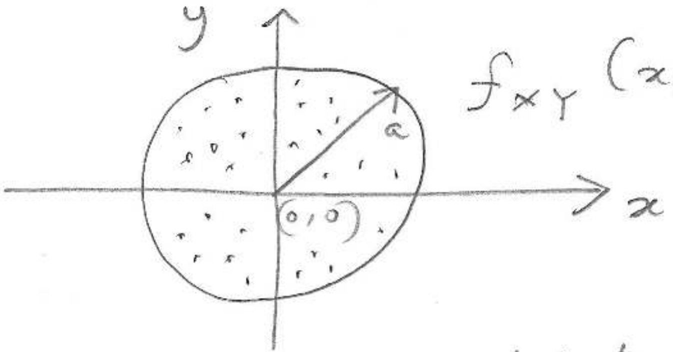
$$|a_n|, |b_n| \leq \frac{1}{\pi n^k} \int_{-\pi}^{\pi} |f^{(k)}(t)|\, dt.$$

PROBLEM 2: This problem has two parts.

(1) Suppose data is uniformly distributed inside a circle of radius a centered at the origin. What can you comment on the set of eigen vectors? Are they unique? (5 pts.)

(2) Consider a sequence of functions $f_n(t) = \frac{t^2+nt}{n}$ $\forall t \in \mathbb{R}$. (a) Examine the pointwise convergence and uniform convergence of $f_n(t)$ on $\mathbb{R}$. (b) If $f_n(t)$ is confined to an interval $[-a, a]$, what can you comment about its uniform convergence and L^2 convergence on the interval $[-a, a]$? Interpret your results graphically. (20 pts.)

1)



$$f_{XY}(x,y) = \begin{cases} \frac{1}{\pi a^2} & x^2+y^2 \le a^2 \\ 0 & \text{else} \end{cases}$$

Fig: 2D density of points inside a circular disc

Intuitively, we can we see that by radial symmetry one can have ∞ pairs of orthogonal eigen vectors ...

$$f_X(x) = \int_{-a\sqrt{1-x^2/a^2}}^{a\sqrt{1-x^2/a^2}} f_{XY}(x,y)\, dy$$

$$= \frac{1}{\pi a^2} \cdot 2a\sqrt{1-x^2/a^2}$$

$$= \begin{cases} \frac{2}{\pi a}\sqrt{1-\frac{x^2}{a^2}} & -a \le x \le a \\ 0 & \text{else.} \end{cases}$$

$$E(x^2) = \int_{-a}^{a} x^2 f_X(x)\, dx = \frac{a^2}{4}$$

$$E(x^2) = E(y^2)$$

($\because$ Symmetric in rotation)

$$E(xy) = \frac{1}{\pi a^2} \int_{x^2+y^2 \le a^2} xy \, dx \, dy$$

Let

$x = a r \cos\theta$, $y = a r \sin\theta$

$$E(xy) = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^1 r^2 a^2 \cos\theta \cdot \sin\theta \; r a^2 \, dr \, d\theta$$

$$= 0$$

The random variable in X and Y are uncorrelated but dependent. Cov (X, Y) is diagonal!

Eigen vectors are not unique and we have ∞ pairs of orthogonal vectors

The eigen value for Cov (X, Y) is $\frac{a^2}{4}$.

NOTE: I gave a full score as long as you realized Cov (X, Y) is diagonal & got a feel of radial symmetry either with explicit calculations or otherwise.

2) Consider $f_n(t) = \dfrac{t^2+nt}{n} \quad \forall\, t \in \mathbb{R}$

(a) Consider the case of $f_n(t)$ convergence on $\mathbb{R}$. We need to look into 2 cases

(i) point wise convergence (ii) uniform convergence

$\lim_{n\to\infty} f_n(t) = f(t) = t$

(i) $|f_n(t) - f(t)| = \dfrac{t^2}{n}$

Now for $\varepsilon > 0$ and $\forall\, t \in R$

$\dfrac{t^2}{n} < \varepsilon \implies$ we can fix $N = \underbrace{\left\lceil \frac{t^2}{\varepsilon} \right\rceil}_{\text{ceil}}$

such that $\dfrac{t^2}{\left\lceil \frac{t^2}{\varepsilon} \right\rceil} \leq \dfrac{t^2}{\frac{t^2}{\varepsilon}+1} < \varepsilon$

Observe that N depends on both t, ε.

So, this is pointwise convergente to $f(t) = t$

(ii) The same steps above imply $\{f_n(t)\}$ <u>does not</u> uniformly converge since N depends on ε and '<u>t</u>'. It does not uniformly converge on $\mathbb{R}$.

(b) Consider the case over $[-a, a]$ i.e., finitely supported

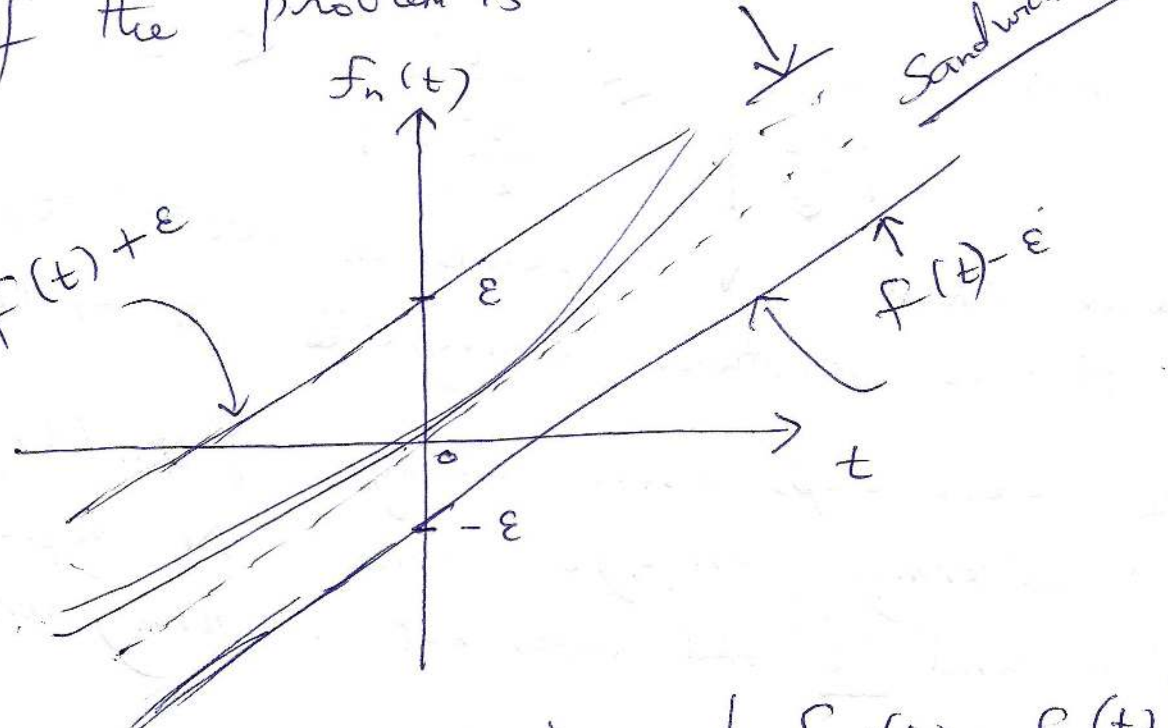
$$|f_n - f| = \frac{t^2}{n} \leq \frac{a^2}{n} \quad \forall t \in [-a, a]$$

So, we can choose $N = \left\lceil \frac{a^2}{\varepsilon} \right\rceil$ such that for $n \geq N$ $|f_n(t) - f(t)| < \varepsilon$

$\forall t \in [-a, a]$

Invoking a theorem we proved in class,

Since a uniformly convergent function $\Rightarrow$ Convergence in L^2, the other part of the problem is solved.



Suppose $\varepsilon > 0$ is given $|f_n(t) - f(t)| < \varepsilon$

$\Rightarrow f(t) - \varepsilon < f_n(t) < f(t) + \varepsilon$

$f_n(t)$ is uniformly close to $f(t)$ is the graphical meaning.

PROBLEM 3: This problem has two parts

(1) Decompose the signal $f(t)$ using the Haar basis. Indicate the signal dimension at each subspace. Sketch the waveforms explicitly at each subspace. If you null out the subspace corresponding to the details at the highest resolution, what is your reconstructed signal in functional form? How much of energy is lost in the recovered signal?

$$f(t) = \begin{cases} 2 & 0 \le t < 0.25 \\ -4 & 0.25 \le t < 0.5 \\ 0 & 0.5 \le t < 0.75 \\ 1 & 0.75 \le t < 1 \end{cases}$$

(20 pts.)

(2) It is observed that a certain signal has a minimum resolution of $\frac{1}{5}$ time units. Suppose we are interested in a wavelet decomposition of the signal, what would be your choice of the scaling function and wavelet using Haar basis? What is the signal dimension in subspace $\mathcal{V}_n$ in this case? (5 pts.)

1) The smallest time is $\frac{1}{4}$ time units. We start with V_2. From the decomposition

$V_2 = V_0 \oplus W_0 \oplus W_1$.

We need the coefficients in each of the above subspaces.

$$f(t) = 2\phi(4t) - 4\phi(4t-1) + \phi(4t-3) \in V_2$$

From class notes,

Now
$$\phi(4t) = \frac{\psi(2t) + \phi(2t)}{2}$$

$$\phi(4t-1) = \frac{\phi(2t) - \psi(2t)}{2}$$

$$\phi(4t-3) = \frac{\phi(2(t-1/2)) - \psi(2(t-1/2))}{2}$$

$$\phi(2t) = \frac{\psi(t) + \phi(t)}{2} \qquad \text{(A)}$$

$$\phi(2t-1) = \frac{\phi(t) - \psi(t)}{2}$$

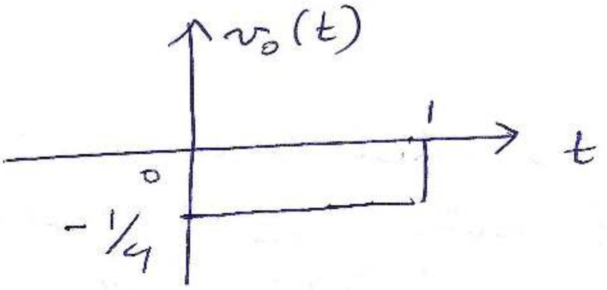
Using (A),

$$f(t) = \underbrace{-\frac{1}{4}\,\phi(t)}_{v_0(t)} \underbrace{-\frac{3}{4}\,\psi(t)}_{w_0(t)} + \underbrace{3\psi(2t) - \frac{1}{2}\psi(2t-1)}_{w_1(t)}$$

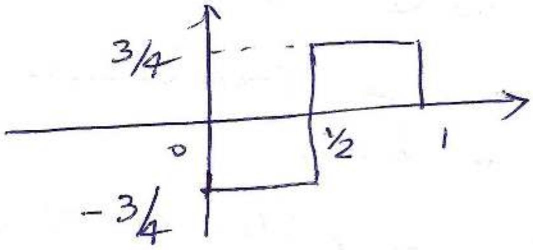
$$\text{Energy in } f(t) = \frac{1}{16} + \frac{9}{16} + \frac{37}{4}\cdot\frac{1}{2} = \frac{84}{16}$$

$\dim(V_0) = 1$ $\quad \dim(W_0) = 1$ $\quad \dim(W_1) = 2$

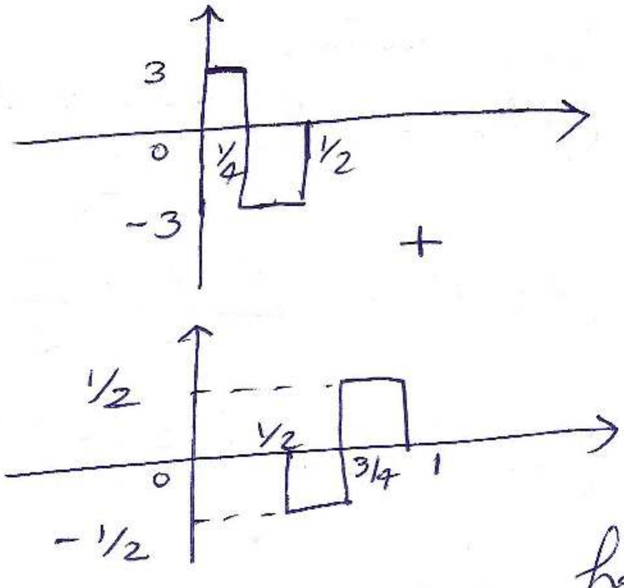
In V_0, we have



In W_0, we have



In W_1, we have



Nulling out signal in W_1, we have

$$\hat{f}(t) = -\frac{1}{4}\,\phi(t) - \frac{3}{4}\,\psi(t).$$

$$\text{Fraction of Energy lost} = \frac{37/8}{84/16} = \frac{74}{84} \simeq 88\,\%.$$

(Looks like most of the energy is in details!)

b) Signal resolution is $\frac{1}{5}$ time units

(i) Either choose $\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$

with a scalar factor $a = 5$ and $\{a^{j/2} \phi(a^j t - k)\}_{k \in \mathbb{Z}}$ as your basis

<u>OR</u>

(ii) Realise $\frac{1}{8} < \frac{1}{5} < \frac{1}{4}$ and go with a decomposition using $\left\{2^{-3/2} \phi(2^3 t - k)\right\}_{k \in \mathbb{Z}}$

In (i) $\dim(V_n) = 5^n$

(ii) $\dim(V_n) = 2^n$

PROBLEM 4: Consider a J stage dyadic decomposition as shown in Figure 2(A). Let the low pass filter $H_0(z)$ and high pass filter $H_1(z)$ be first order FIR filters derived from the Haar basis. The filters are normalized to unit energy. This forms the analysis stage. We would like to have an equivalent representation as in Figure 2(B) with increasing decimation rates i.e., $D_0 \leq D_1 \leq ... \leq D_n$ as we progress from the top branch to the bottom branch in Figure 2(B) with one-to-one correspondence to branches in Figure 2(A).

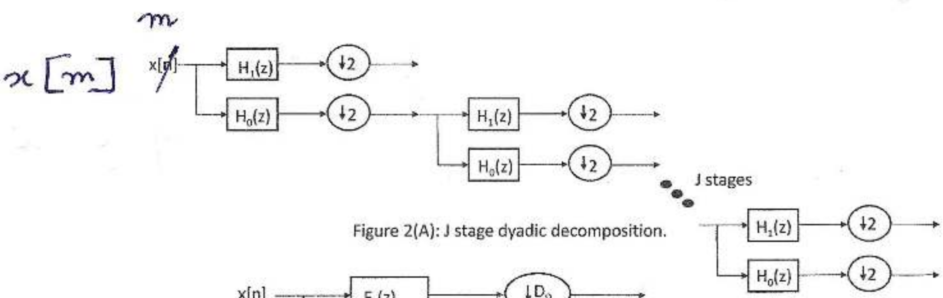


Figure 2(A): J stage dyadic decomposition.

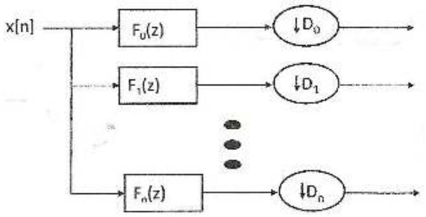


Figure 2(B): Equivalent representation of Figure 2(A).

(1) What is n in Figure 2(B) in terms of J? Determine all the filters $F_i(z)$, $0 \leq i \leq n$ in terms $H_0(z)$ and $H_1(z)$. What are the values of the decimation rates D_i in Figure 2(B)? What frequency band does each branch correspond to in terms of normalized angular frequencies? (10 pts.)

(2) Obtain the architecture for the synthesis stage mirroring to the form in Figure 2(B) using upsamplers and synthesis filters. Explicitly compute the transfer functions of the corresponding synthesis filters indicating the filter orders. (10 pts.)

(3) What can you say about $\sum_{i=0}^{n} \frac{1}{D_i}$? (5 pts.)

(1) For a J stage decomposition, we have $J+1$ branches $\Rightarrow$ $n = J$

(a) $F_0(z) = H_1(z)$, $D_0 = 2$

(b) $\rightarrow \boxed{H_0(z)} \rightarrow (\downarrow 2) \rightarrow \boxed{H_1(z)} \rightarrow (\downarrow 2) \rightarrow$

$\equiv \rightarrow \boxed{H_0(z)} \rightarrow \boxed{H_1(z^2)} \rightarrow (\downarrow 4)$ (By Nobel identities)

$$F_1(z) = H_0(z)\, H_1(z^2), \quad D_1 = 4$$

Continuing with this line of thought,

$$F_k(z) = H_1\left(z^{2^k}\right) \prod_{j=1}^{k-1} H_0\left(z^{2^{j-1}}\right)$$

$$1 \le k \le n-1$$

For the last branch,

$$F_n(z) = \prod_{j=0}^{n-1} H_0\left(z^{2^{j-1}}\right)$$

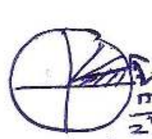
$$D_n = 2^n$$

$$D_k = 2^{k+1} \qquad 0 \le k \le n-1.$$

$H_0(z)$ is a LPF & $H_1(z)$ is a HPF

$F_k(z)$ corresponds to the band

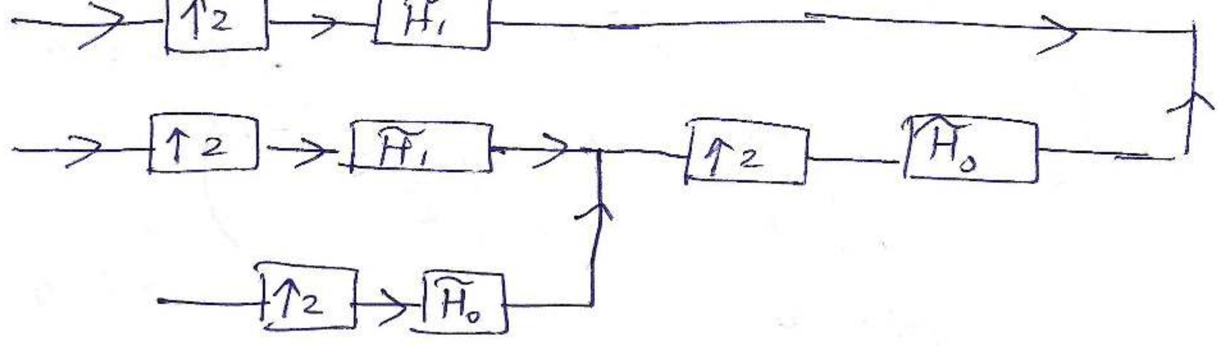
$$\left[-\frac{\pi}{2^k}, -\frac{\pi}{2^{k+1}}\right] \cup \left[\frac{\pi}{2^k}, \frac{\pi}{2^{k+1}}\right]$$



F_n is $\left[-\frac{\pi}{2^n}, \frac{\pi}{2^n}\right]$

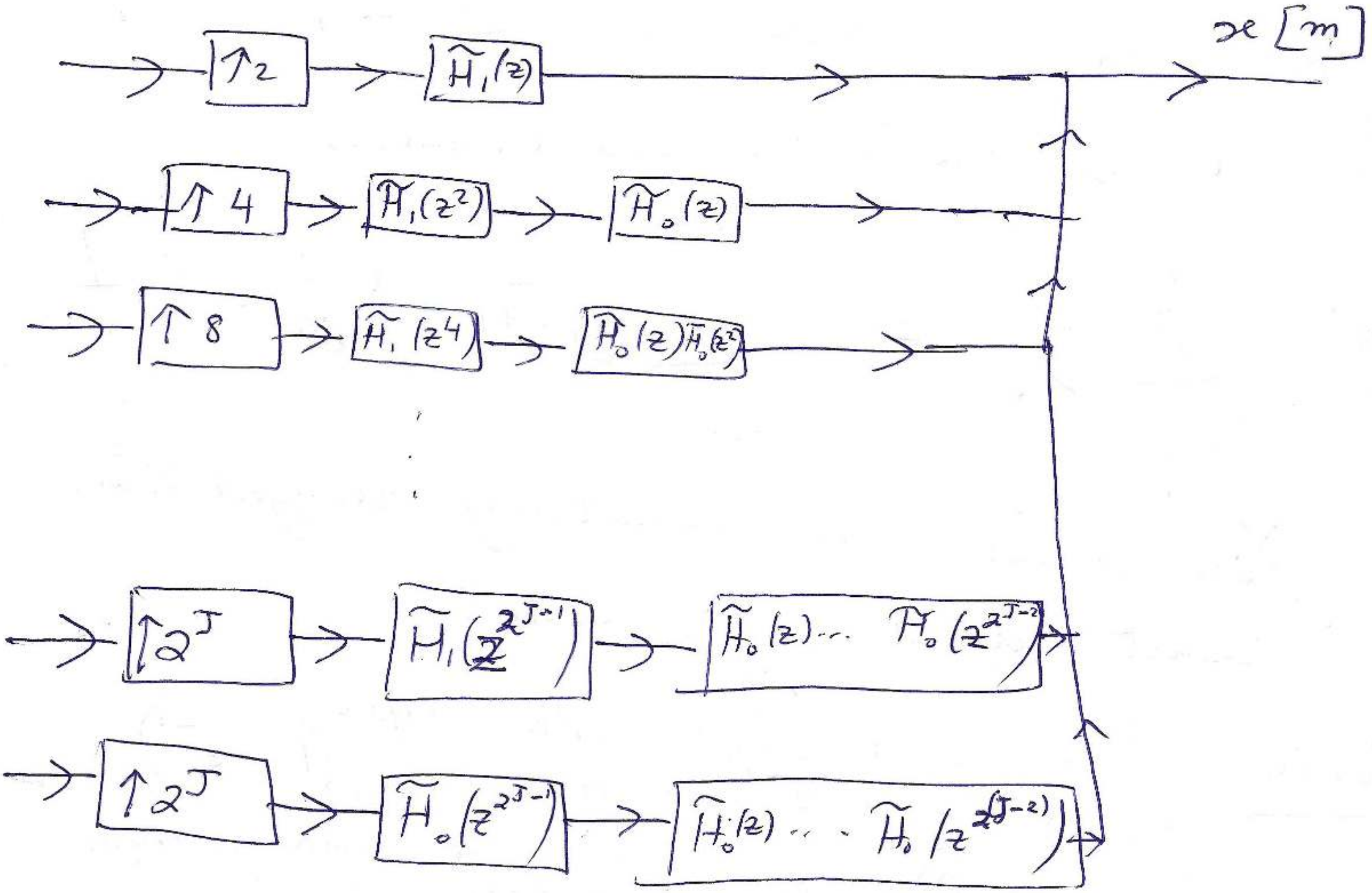
(I am okay if you just did a disc partition diagram with +ve frequencies)

(2) We follow a similar path mirroring what we did in the analysis stage.



∴ J stages

Following the analysis step, our equivalent structure is



$$\tilde{F}_k(z) = \begin{cases} \tilde{H}_1(z) & k = 0 \\ \tilde{H}_1(z^{2^k}) \prod_{j=1}^{k-1} \tilde{H}_0(z^{2^{j-1}}) & 1 \le k \le n-1 \\ \prod_{j=0}^{n-1} \tilde{H}_0(z^{2^{j-1}}) & k = n \end{cases}$$

<u>The filter orders</u>

For $k = 0$, order $= 1$

$1 \le k \le n-1$ order $= 2^{k+1} - 1$

$k = n$ order $= 2^n - 1$

As you see, the order also increases . . .

(3) $$\sum_{i=0}^{n} \frac{1}{D_i} = \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} + \frac{1}{2^n} = 1$$

You should have intuitively realized this result without any math!

<u>NOTE</u>: You could start with simple $H_0(z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$ $H_1(z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$ after normalizing the energy & making them causal or do it as is per class notes.