**INDIAN INSTITUTE OF SCIENCE**
**E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING**
**HOME WORK #5 - SOLUTIONS, FALL 2015**

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**Problem 1.** 7.2.3 from Moon & Stirling

**Solution.** As derived in the class, SVD of $\mathbf{A}$ is

$$\begin{aligned} \mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H &= \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix} \begin{bmatrix}\boldsymbol{\Sigma}_1 & \underline{0} \\ \underline{0}^T & \mathbf{0}\end{bmatrix} \begin{bmatrix}\mathbf{V}_1^H \\ \mathbf{V}_2^H\end{bmatrix} \\ &= \mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{V}_1^H \\ \Longrightarrow \mathbf{A}^H &= \mathbf{V}_1\boldsymbol{\Sigma}_1^T\mathbf{U}_1^H \end{aligned}$$

From the above equations, the four fundamental sub-spaces related to the matrix $\mathbf{A}$ are
a) Range space (column space) of $\mathbf{A}$:

$$\begin{aligned} \mathcal{R}(\mathbf{A}) &= \{\mathbf{A}\underline{x} \mid \underline{x} \in \mathbb{C}^n\} \\ &= \left\{\mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{V}_1^H\underline{x} \mid \underline{x} \in \mathbb{C}^n\right\} \\ &= \{\mathbf{U}_1\underline{\hat{x}} \mid \underline{\hat{x}} \in \mathbb{C}^r\} \\ &= \operatorname{Span}(\mathbf{U}_1) \end{aligned}$$

b) Range space (column space) of $\mathbf{A}^H$:

$$\mathcal{R}\left(\mathbf{A}^H\right) = \operatorname{Span}(\mathbf{V}_1)$$

c) Null space of $\mathbf{A}$: From the theorem proved in the class,

$$\mathcal{N}(\mathbf{A}) = \left[\mathcal{R}\left(\mathbf{A}^H\right)\right]^\perp = \operatorname{Span}(\mathbf{V}_2)$$

b) Null space of $\mathbf{A}^H$: From the theorem proved in the class,

$$\mathcal{N}\left(\mathbf{A}^H\right) = [\mathcal{R}(\mathbf{A})]^\perp = \operatorname{Span}(\mathbf{U}_2)$$

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**Problem 2.** 7.2.4 from Moon & Stirling

**Solution.** Given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 6 & 7 & 2 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 48 \\ 30 \end{bmatrix}.$$

We need to find least square solution for $\mathbf{A}\underline{x} = \underline{b}$.

Since rank of $\mathbf{A}$ is 2, $\underline{b}$ lies in $\mathcal{R}(\mathbf{A})$. Therefore, the projection of $\underline{b}$ onto $\mathcal{R}(\mathbf{A})$ is $\underline{b}$ itself.

The SVD of $\mathbf{A}$ is

$$\mathbf{A} = \underbrace{\begin{bmatrix} 0.6636 & 0.7480 \\ 0.7840 & -0.6636 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 11.5913 & 0 & 0 & 0 \\ 0 & 5.8001 & 0 & 0 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} 0.4445 & 0.6808 & 0.4153 & 0.4081 \\ -0.5575 & -0.2851 & 0.4160 & 0.6954 \\ 0.4661 & -0.3267 & -0.5573 & 0.6045 \\ 0.5237 & -0.5904 & 0.5864 & -0.1823 \end{bmatrix}}_{\mathbf{V}^H}.$$

The least squares inverse is

$$\mathbf{A}^\dagger = \mathbf{V}\mathbf{\Sigma}^\dagger\mathbf{U}^H$$

where

$$\mathbf{\Sigma}^\dagger = \begin{bmatrix} \frac{1}{11.5913} & 0 \\ 0 & \frac{1}{5.8001} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\mathbf{A}^\dagger = \begin{bmatrix} -0.0465 & 0.0925 \\ 0.0022 & 0.0765 \\ 0.0774 & -0.0208 \\ 0.1084 & -0.0491 \end{bmatrix}.$$

The least square solution is

$$\underline{\hat{x}} = \mathbf{A}^\dagger \underline{b} = \begin{bmatrix} 0.5442 \\ 2.4027 \\ 3.0929 \\ 3.7301 \end{bmatrix}$$

The $l_2$ norm of the solution is

$$\|\underline{\hat{x}}\| = 5.4359 < 5.4772 = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\|.$$

Since the equation $\mathbf{A}\underline{x} = \underline{b}$ has infinite solutions, a constraint is generally enforced to identify a suitable unique solution. The choice of this constraint on the solution depends on the problem:

A least squares solution is desired if the samples in $\underline{b}$ are erroneous. ■

**Problem 3.** 7.7.13 from Moon & Stirling

**Solution.** We have $\underline{y} \in \mathcal{R}\left(\tilde{\mathbf{V}}\right)$ with $y_{m+1} = -1$ and $\underline{x} = \tilde{\mathbf{I}}\underline{y}$ where

$$\tilde{\mathbf{I}} = \begin{bmatrix} \mathbf{I}_m & \underline{0} \\ \underline{0}^T & 0 \end{bmatrix}.$$

Let the dimension of $\tilde{\mathbf{V}}$ is $(m+1) \times p$. $p$ is the number of times the smallest singular value of $\mathbf{A}$ is repeated.
We can write $\underline{y} = \tilde{\mathbf{V}}\underline{a}$ where $\underline{a}$ is vector whose dimension is $p$.
Our goal is to find $\underline{y}$ i.e., to find $\underline{a}$ such that
a) $\|\underline{x}\|^2 = \left\|\tilde{\mathbf{I}}\underline{y}\right\|^2$ is minimized
b) the constraint $y_{m+1} = -1$ is satisfied i.e., $\underline{u}^T\underline{y} + 1 = 0$ where

$$\underline{u}^T = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{1\times(m+1)}.$$

We solve this problem using Lagrange multiplier $\lambda$ by minimizing the cost function given by

$$\begin{aligned}
C(\underline{a}, \lambda) &= \left\|\tilde{\mathbf{I}}\underline{y}\right\|^2 + 2\lambda\left(\underline{u}^T\underline{y} + 1\right) \\
&= \left\|\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a}\right\|^2 + 2\lambda\left(\underline{u}^T\tilde{\mathbf{V}}\underline{a} + 1\right) \\
&= \left(\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a}\right)^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a} + 2\lambda\left(\underline{u}^T\tilde{\mathbf{V}}\underline{a} + 1\right) \\
&= \underline{a}^H\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a} + 2\lambda\left(\underline{u}^T\tilde{\mathbf{V}}\underline{a} + 1\right) \\
&= \underline{a}^H\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a} + 2\lambda\left(\underline{u}^T\tilde{\mathbf{V}}\underline{a} + 1\right) \quad \left(\tilde{\mathbf{I}}^H\tilde{\mathbf{I}} = \tilde{\mathbf{I}}\right)
\end{aligned}$$

$$\frac{\partial C}{\partial \underline{a}^H} = 2\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a} + 2\lambda\tilde{\mathbf{V}}^H\underline{u} = \underline{0}. \tag{1}$$

$$\frac{\partial C}{\partial \lambda} = \underline{u}^T\tilde{\mathbf{V}}\underline{a} + 1 = 0. \tag{2}$$

From (1), we have

$$\begin{aligned}
\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\underline{a} &= -\lambda\tilde{\mathbf{V}}^H\underline{u} \\
\implies \underline{a} &= -\lambda\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\underline{u}.
\end{aligned} \tag{3}$$

Using (3) in (2), we have

$$\begin{aligned}
\lambda\underline{u}^T\left(\tilde{\mathbf{V}}\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\right)\underline{u} &= 1 \\
\implies \lambda &= \frac{1}{\underline{u}^T\left(\tilde{\mathbf{V}}\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\right)\underline{u}} \\
\implies \underline{a} &= -\frac{\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\underline{u}}{\underline{u}^T\left(\tilde{\mathbf{V}}\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\right)\underline{u}}
\end{aligned}$$

Therefore, the desired solution is

$$\underline{y} = \tilde{\mathbf{V}}\underline{a} = -\frac{\tilde{\mathbf{V}}\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\underline{u}}{\underline{u}^T\left(\tilde{\mathbf{V}}\left(\tilde{\mathbf{V}}^H\tilde{\mathbf{I}}\tilde{\mathbf{V}}\right)^{-1}\tilde{\mathbf{V}}^H\right)\underline{u}}.$$

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