Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

Instructor: Shayan G. Srinivasa

Mid Term Exam#1, Fall 2016

Name and SR.No:

Instructions:

- Only four A4 pages/sheets of paper with written notes are allowed.
- The time duration is 3 hrs.
- There are four main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
Total points	

SOLUTIONS:

Prepared by: Shounak Roy, Chaitanya Kumar Matcha

Scribed by: Harshitha Srinivas

PROBLEM 1: This problem has 3 parts.

(1) (a) Is the inverse of a causal LTI system causal? Justify. (b) Is a finite duration signal always stable? Justify (5) pts.

(2) Let $\mathcal{V}$ be a vector space. Suppose $\mathcal{W}_1$ and $\mathcal{W}_2$ are subspaces of $\mathcal{V}$. Show that $\mathcal{W}_1+\mathcal{W}_2$ is a subspace of $\mathcal{V}$ that contains $\mathcal{W}_1$ and $\mathcal{W}_2$. (10 pts.)

(3) Consider the space $\mathcal{V}$ spanned by the vectors $\mathbf{v}_1 = (1\ 2\ 1)^{\mathrm{T}}$, $\mathbf{v}_2 = (1\ 0\ 1)^{\mathrm{T}}$ and $\mathbf{v}_3 = (0\ -2\ 0)^{\mathrm{T}}$. Obtain the basis and dimension of $\mathcal{V}$ and $\mathcal{V}^{\perp}$. (10 pts.)

Solution:

(Part 1a)

Consider the case $H(z){=}z^{-1}$, $H^{-1}(z) = z$ (anti-causal). consider the case $H(z) = \frac{1}{1-z^{-1}}$, $H^{-1}(z) = 1 + z^{-1} + ...\infty$(causal). One can have it causal or anti-causal.

In general, for any rational transfer function

$$H(z) = \frac{\sum_{k=0}^{p} b_k z^{-k}}{1 + \sum_{k=1}^{q} a_k z^{-k}}, \tag{1}$$

one can deduce condition for inverses to be causal depending on coefficients b_k's and a_k's. The trouble is seen readily when $b_0 = 0$.

(Part 1b)

$\sum_{k=0}^{N-1} |x(k)|^2 < \infty \Longrightarrow$ stable provided $|x\,(k)| < \infty \forall k$.

(Part 2)

To prove $W_1 + W_2$ is a subspace of V, we need to show following properties. All other properties of a vector space trivially hold true.

(a) Identity: $0 \in W_1$,$0 \in W_2 \Longrightarrow 0 = 0 + 0 \in W_1 + W_2$ (definition of $W_1 + W_2$).

(b) Scalar multiplication Suppose $a{\in}R$ is an arbitrary real and $x{\in}W_1{+}W_2$. By definition $\exists\, x_1{\in}W_1$ and $\exists\, x_1{\in}W_2$ s.t. $x = x_1{+}x_2$, therefore $ax = a(x_1{+}x_2){=}\ ax_1{+}ax_2$. Since $x_1 \leq W_1$,$ax_1{\in}W_1$. Similarly $x_2 \in W_2 \implies ax_2 \in W_2$. Therefore $ax \in W_1 + W_2$by definition.

(c) Closure under addition

Let $x, y \in W_1 + W_2$be arbitrary vectors in $W_1 + W_2$

$x_1, x_2 \in W_1$and $y_1, y_2 \in W_2$ s.t. $x = x_1 + y_1$

$x_1 + x_2 \in W_1$and $y_1 + y_2 \in W_2$ s.t. $y = x_2 + y_2$

Therefore $x + y{=}\ (x_1 + y_1) + (x_2 + y_2)$

$x + y \in W_1 + W_2.$

(d) What remains to show is $W_1 \subseteq W_1 + W_2$, $W_1 \subseteq W_1 + W_2$ and $W_1 + W_2 \subseteq V$.

Suppose $x_1 \in W_1$ and $0 \in W_2$. Now $x = x + 0{\in}W_1 + W_2$(definition). But, $x \in W_1$was any vector. Every element of W_1 is contained in $W_1 + W_2$. Similarly choosing $x \in W_2$ and $0\ {\in}W_1$, we infer the same. Therefore $W_1 \subseteq W_1 + W_2$; and $W_2 \subseteq W_1 + W_2$.

Suppose $x \in W_1 + W_2$. Then $\exists x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$. Now $W_1 \subseteq V$ and $W_2 \subseteq V \implies x_1, x_2 \in V$. Therefore $x = x_1 + x_2 \in V$. Therefore, $W_1 + W_2 \subseteq V$.

(Part 3)

$$\underset{v_1}{(1, 2, 1)^T} \quad \underset{v_2}{(1, 0, 1)^T} \quad \underset{v_3}{(0, -2, 0)^T}$$

Clearly $v_3 = v_2 - v_3 \implies v_1, v_2, v_3$ are linearly independent.

But v_2, v_3 are orthogonal $\Longrightarrow$ linearly independent and span V.

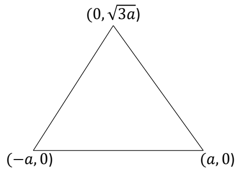
Therefore, basics for V is $\left\{\frac{v_2}{\sqrt{2}}, \frac{v_3}{2}\right\}$ and $\dim(V) = 2$.
Let $u = \frac{c}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}^T + c_2\begin{pmatrix}0 & -1 & 0\end{pmatrix}^T \in V$
$u = \begin{pmatrix}\frac{c_1}{\sqrt{2}} & -c_2 & \frac{c_1}{\sqrt{2}}\end{pmatrix}^T \in V$
Let $v = \begin{pmatrix}a & b & c\end{pmatrix}^T \in V^{\perp}$
$\langle u, v\rangle = 0 \Longrightarrow \frac{ac_1}{\sqrt{2}} - bc_2 + \frac{cc_1}{\sqrt{2}} = 0.$
$a + c = 0$ and $b = 0$ is admissible.
So a basis for $V^{\perp}$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $\dim(V^{\perp}) = 1$.
At least you must have realized that $\dim(V) + \dim(V^{\perp}) = 3$ and got $\dim(V^{\perp})$.

PROBLEM 2: This problem has 2 parts.

(1) Suppose the joint probability mass function (pmf) P_{XY} is uniform over all the three corners of an equilateral triangle whose base has vertices at $(-a, 0)$ and $(a, 0)$. Obtain the marginal pmfs. Are the random variables (a) independent (b) correlated? (10 pts.)

(2) Consider the random process $S(t) = A\cos(\omega t) + B\sin(\omega t)$, where ω is a constant and A and B are random variables. (a) What is the necessary condition for this process to be stationary? (b) If A and B are uncorrelated with equal variance, then $S(t)$ is wide sense stationary. Justify if the statement is correct. (15 pts.)

Solution



(Part 1a) You could also have $(a, -\sqrt{3}a)$ (flipped vertex) - it does not matter.

p_{xy}	x=-a	x=0	x=a
y=0	1/3	0	1/3
y=$\sqrt{3}a$	0	1/3	0

$p(X=0) = \sum_y p_{XY}(X=0, Y=y) = \frac{1}{3}$. Similarly $p_X(X=a) = \frac{1}{3}$ and $p_X(X=-a) = \frac{1}{3}$.

$p(Y=0) = \sum_x p_{XY}(X=x, Y=0) = \frac{2}{3}$, and $p_y(Y=\sqrt{3}a) = \frac{1}{3}$

Examine $p_{XY}(X=0, Y=0) = 0 \neq p_X(X=0)p_Y(X=0) = \frac{2}{9}$.

Therefore, RVs are not independent statistically.

(Part 1b)
$E(X) = \sum_x x p_X(X=x) = \frac{1}{3}(-a) + \frac{1}{3}(0) + \frac{1}{3}(a) = 0$, $E(Y) = \sum_y y p_y(Y=y) = \frac{2}{3}(0) + \frac{1}{3}(\sqrt{3}a) = \frac{a}{\sqrt{3}}$, $E(XY) = \sum_{xy} xy P_{XY}(X=x, Y=y) = \frac{1}{3}(0) + \frac{1}{3}(0) + \frac{1}{3}(0) = 0$
$E(XY) - E(X)E(Y) = 0$ is satisfied. They are uncorrelated.

(Part 2a)
Since we need only a sufficient condition, we can enforce any property of a stationary process to obtain a sufficient condition.

Approach 1:
A stationary process is a WSS. Therefore, any or all conditions in Part 2b is are sufficient conditions.

Approach 2:
We can further enforce conditions of shift in variance on higher order moments:
a) $E(s^k(t))$ must be independent of $'t'$ for $k = 1, 2, 3, \cdots$. This would result conditions on $E[A^l B^{k-l}]$ $l = 0, 1, \cdots k$. However, the derivation of these conditions use the orthogonality of $\sin(k\omega t)$ and $\cos(k\omega t)$. The derivations are not trivial.

(Part 2b)
For a WSS process, mean$(s(t))$ is a constant and autocorrelation must depend only on the lag.

$$\begin{aligned}\mu(t) &= E(s(t)) \\ &= E((A\cos(\omega t) + B\sin(\omega t)) \\ &= E(A)\cos(\omega t) + E(B)\sin(\omega t)\end{aligned}$$

$$
\begin{aligned}
R_s(t, t+\tau) &= E(s(t)s(t+\tau)) \\
&= E((A\cos(\omega t) + B\sin(\omega t))(A\cos(\omega(t+\tau) + B\sin(\omega(t+\tau))) \\
&= \frac{1}{2}[E(A^2) + E(B^2)]\cos(\omega\tau) + \frac{1}{2}[E(A^2) - E(B^2)]\cos(2\omega t - \omega\tau) \\
&+ \frac{1}{2}E(AB) * 2 * \sin(2\omega t + \omega\tau)
\end{aligned}
$$

Necessary condition: In the above $\cos(\omega t)$ and $\sin(\omega t)$ are linearly independent. Therefore, for $\mu(t)$ to be independent of t, we need $E(A) = E(B) = 0$. Similarly, $\cos(2\omega t - \omega\tau)$ and $\sin(2\omega t + \omega\tau)$ are linearly independent. Therefore, for $R_s(t + t + \tau)$ to only depend on τ, $E(AB) = 0$ and $E(A^2) = E(B^2) = k$.

Sufficient condition: If $E(A) = E(B) = 0$, then $\mu(t) = 0$. If $E(AB) = 0$ and $E(A^2) = E(B^2) = k$, then $R_s(t, t+\tau) = kcos(\omega t)$. Therefore the process is WSS.

PROBLEM 3: This problem has 2 parts.

(1) If the low pass filter in a QMF bank is linear phase, the overall transfer function between the reconstructed output and input is guaranteed to be linear phase. Examine if this statement is true/false. Justify. (10 pts.)

(2) Suppose the low pass filter in a two-channel QMF bank is given by $H_0(z) = 2+6z^{-1}+z^{-2}+5z^{-3}+z^{-5}$, obtain a set of stable synthesis filters for perfect recovery. Sketch the polyphase implementation schematic. (15 pts.)

Solution:

(Part 1)

From polyphase decomposition, for a QMF bank,

$$\begin{aligned}
H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2), \\
H_1(z) &= E_0(z^2) - z^{-1}E_1(z^2), \\
T(z) &= \frac{1}{2}[H_0^2(z) - H_1^2(z)] = 2z^{-1}E_0(z^2)E_1(z^2).
\end{aligned}$$

Suppose $H_0(z)$ is linear phase, i.e., $h_0[n] = h_0[N-n] \implies H_0(z) = z^{-N}H_0(z)$, where N is the order, let us consider 2 cases:

(a) N+1 odd :

$$\begin{aligned}
H_0(z) &= a_0 + a_1z^{-2} + ... + a_{\frac{N-1}{2}}z^{\frac{-(N-1)}{2}} + ... + a_0z^{-N}, \\
E_0(z^2) &= a_0 + a_2z^{-2} + ... + a_2z^{-(N-2)} + ... + a_0z^{-N}, \\
E_1(z^2) &= a_1 + a_3z^{-2} + ... + a_1z^{-(N-1)}.
\end{aligned}$$

$E_0(z^2)$ and $E_1(z^2)$ are clearly linear phase filters. Therefore, $2z^{-1}E_0(z^2)E_1(z^2)$ is a linear phase.

(b) N+1 even:

$$\begin{aligned}
H_0(z) &= a_0 + a_1z^{-2} + ... + a_0z^{-N}, \\
E_0(z^2) &= a_0 + a_2z^{-2} + ... + a_1z^{-(N-1)}, \\
E_1(z^2) &= a_1 + a_3z^{-2}...... + a_0z^{-(N-1)}.
\end{aligned}$$

$$\begin{aligned}
&\implies E_0(z^2) = z^{-(N-1)}E_1(z^{-2}) \\
&\text{and } E_1(z^2) = z^{-(N-1)}E_0(z^{-2}).
\end{aligned}$$

$$\implies E_0(z^2)E_1(z^2) = z^{-2(N-1)}E_0(z^{-2})E_1(z^{-2}) \implies \text{linear phase.}$$

Therefore, $z^{-1}E_0(z^2)E_1(z^2)$ is Linear phase.

(Part 2)

For perfect recovery,

$$H_0(z)F_0(z) + H_1(z)F_1(z) = cz^{-n_0},$$

Writing

$$\begin{aligned}
H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2) \\
H_1(z) &= E_0(z^2) - z^{-1}E_1(z^2),
\end{aligned}$$

$$\implies E_0(z^2)[F_0(z)+F_1(z)] + z^{-1}E_1(z^2)[F_0(z)-F_1(z)] = cz^{-n_0}.$$

Let

$$F_0(z)+F_1(z) = \frac{c}{2}\frac{z^{-(n_0)}}{E_0(z^2)}$$
$$F_0(z)-F_1(z) == \frac{c}{2}\frac{z^{-(n_0+1)}}{E_1(z^2)}.$$

This gives,

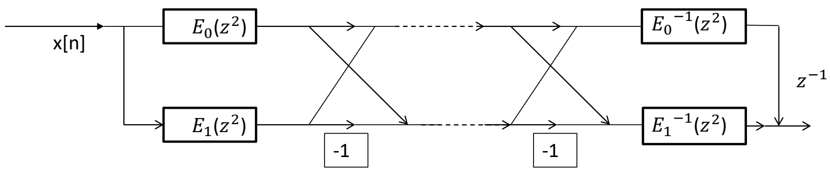
$$F_0(z) = \frac{1}{4}cz^{-(n_0-1)}[\frac{z^{-1}}{E_0(z^2)} + \frac{1}{E_1(z)^2}],$$
$$F_1(z) = \frac{1}{4}cz^{-(n_0-1)}[\frac{z^{-1}}{E_0(z^2)} + \frac{1}{E_1(z)^2}].$$

We have $E_0(z^2) = 2 + z^{-2}$, $E_1(z^2) = 6 + 5z^{-2} + z^{-4}$. Since $\frac{1}{E_0(z^2)}$ and $\frac{1}{E_1(z^2)}$ are stable, the reconstruction filters are also stable.

Choose $n_0 = 1$ and $c = 4$, we have the polyphase representation of $F_0(z)$ and $F_1(z)$ as

$$F_0(z) = z^{-1}E_1^{-1}(z^2) + E_0^{-1}(z^2),$$
$$F_1(z) = z^{-1}E_1^{-1}(z^2) + E_0^{-1}(z^2).$$

The polyphase implementation is shown below:



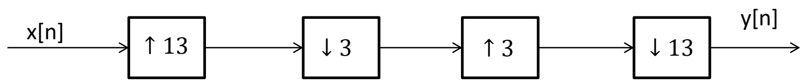
PROBLEM 4: This problem has 2 parts.

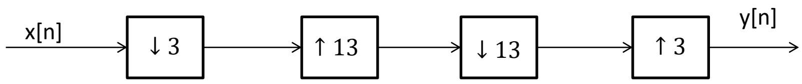
(1) Suppose a discrete time signal $x[n]$ is first upsampled by 13 followed by downsampling and upsampling by 3 and downsampling by 13 in the process of sampling rate conversions without any filtering operations in-between. Obtain the frequency domain response at the output after all your simplifications. (5 pts.)

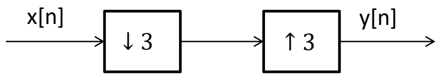
(2) We need an efficient sampling rate conversion from 32 Ksamples/s to 48 Ksamples/s. From first principles, derive a fully efficient multirate architecture with all associated filters. Sketch the schematic of your multirate system. (20 pts.)

Solution:

(Part 1)



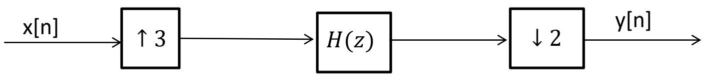


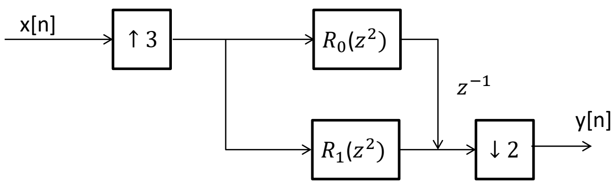


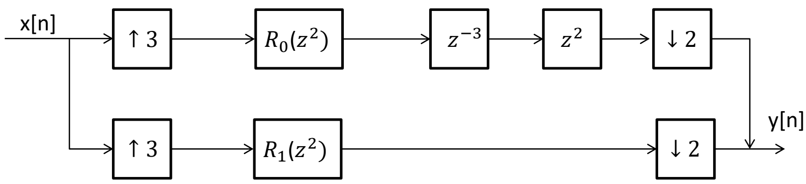
$$Y(z)=\frac{1}{3}[x(z)+x(zW_3)+x(zW_3^2)],\quad W_3=e^{-j\frac{2\Pi}{3}}.$$

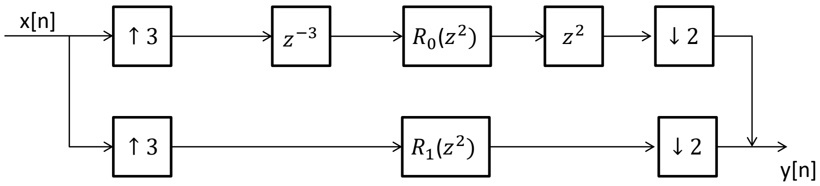
(Part 2)

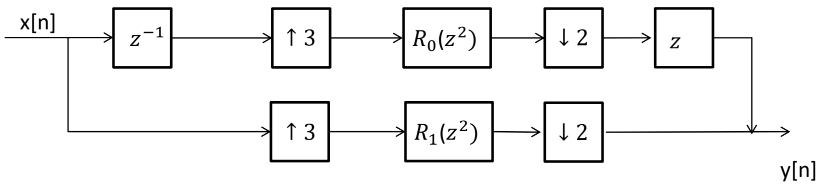
Consider rate is$\frac{48k}{32k}=\frac{3}{2}=1.5$ we need an architecture of the form

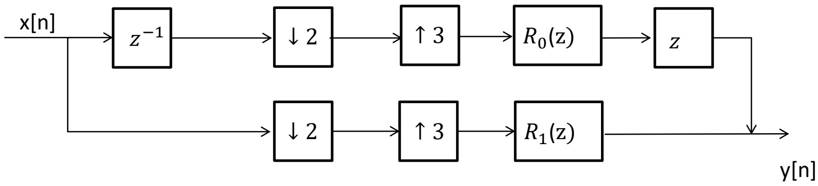






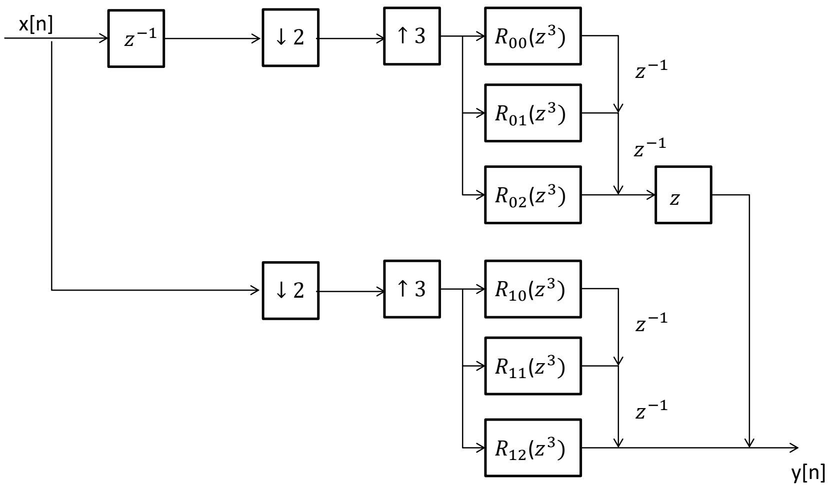


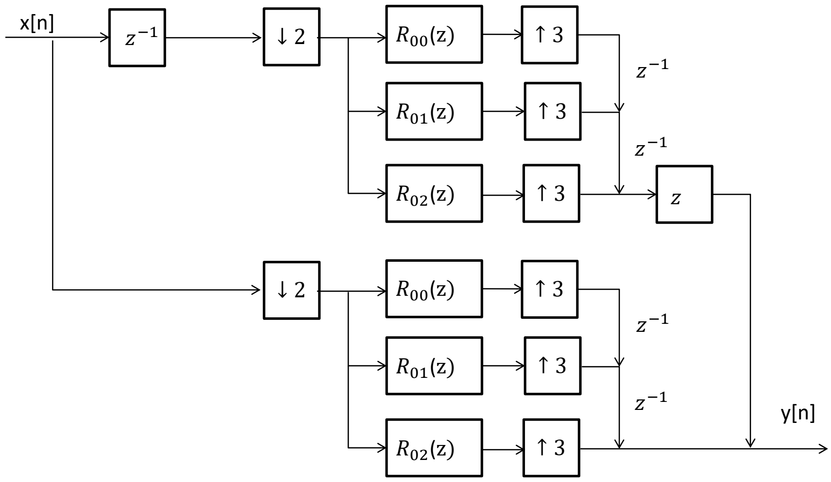




Using type 2 polyphase decomposition again,

$$R_0(z) = R_{02}(z^3) + z^{-1}R_{01}(z^3) + z^{-2}R_{00}(z^3),$$
$$R_1(z) = R_{12}(z^3) + z^{-1}R_{11}(z^3) + z^{-2}R_{10}(z^3).$$





x[n]
z^{-1}
↓ 2
R_{00}(z)
↑ 3
R_{01}(z)
R_{02}(z)
z^{-1}
z
y[n]