E9-252: Mathematical Methods and Techniques in Signal Processing
Homework 1 Solutions

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(P. P. Vaidyanathan, 3.1)
$H(z)$ is given to be a linear phase filter. To prove $G(z)$ a linear phase filter we just have to show $g(n) = g(N-n)$. We have,

$$g(n) = (-1)^M \delta(n-M) - (-1)^n h(n), \tag{1}$$

with $M = \frac{N}{2}$.
(Part a) We have,

$$\begin{aligned} g(N-n) &= (-1)^M \delta(N-n-M) - (-1)^{N-n} h(N-n), \\ &= (-1)^M \delta(M-n) - (-1)^{N-n} h(n), \ [\because h(n) \text{ is a linear phase filter}] \end{aligned} \tag{2}$$

Also we have, $\delta(M-n) = \delta(n-M)$. Thus, we get,

$$g(N-n) = g(n),$$

which proves $g(n)$ is a linear phase FIR filter.

(Part b) Using $h(n) = h(N-n)$, we have

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}+1}^{N} h(n)\, e^{-j\omega n} + h\left(\frac{N}{2}\right) e^{-j\omega \frac{N}{2}} \\
&= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n'=0}^{\frac{N}{2}-1} h(N-n')\, e^{-j\omega\left(N-n'\right)} + h\left(\frac{N}{2}\right) e^{-j\omega \frac{N}{2}} \qquad (n' - N - n) \\
&= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n'=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega(N-n)} + h\left(\frac{N}{2}\right) e^{-j\omega \frac{N}{2}} \\
&= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left(e^{-j\omega n} + e^{-j\omega(N-n)}\right) + h\left(\frac{N}{2}\right) e^{-j\omega \frac{N}{2}} \\
&= e^{-j\omega \frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left(e^{-j\omega\left(n-\frac{N}{2}\right)} + e^{j\omega\left(n-\frac{N}{2}\right)}\right) + h\left(\frac{N}{2}\right) e^{-j\omega \frac{N}{2}} \\
H\left(e^{j\omega}\right) &= e^{-j\omega \frac{N}{2}} \underbrace{\left[2 \sum_{n=0}^{\frac{N}{2}-1} h(n) \cos\left(\omega\left(n - \frac{N}{2}\right)\right) + h\left(\frac{N}{2}\right)\right]}_{H_R(\omega)\ \text{real valued}}
\end{aligned}
$$

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= e^{-j\omega \frac{N}{2}} H_r(\omega) \\
\implies H\left(-e^{j\omega}\right) &= (-1)^{\frac{N}{2}}\, e^{-j\omega \frac{N}{2}} H_R(\omega + \pi)
\end{aligned}
$$

We have,

$$
G\left(e^{j\omega}\right) = (-1)^M e^{-j\omega M} - H\left(-e^{j\omega}\right). \tag{3}
$$

$$
G\left(e^{j\omega}\right) = (-1)^{\frac{N}{2}} e^{-j\omega \frac{N}{2}} - (-1)^{\frac{N}{2}} e^{-j\omega \frac{N}{2}} H_R(\omega + \pi) \tag{4}
$$

$$
G\left(e^{j\omega}\right) = (-1)^{\frac{N}{2}} e^{-j\omega \frac{N}{2}} \left[1 - H_R(\omega + \pi)\right]. \tag{5}
$$

The amplitude response of $H(e^{j\omega})$ is given as follows

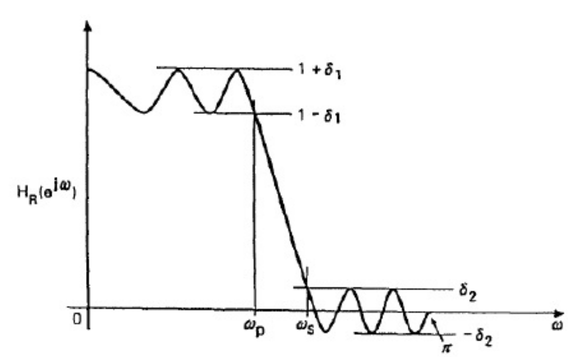
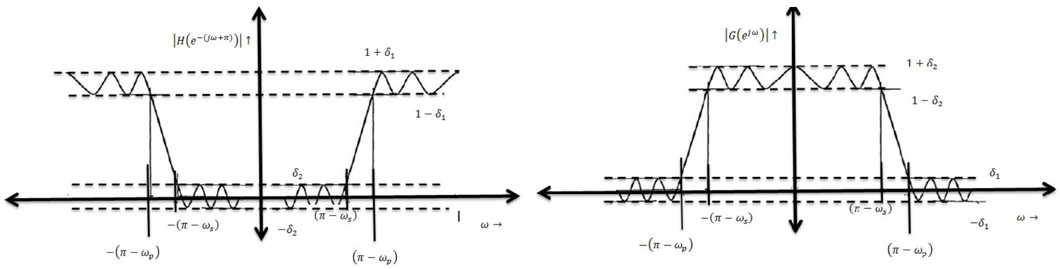


Figure 1: Amplitude response of $|H(e^{j\omega})|$

Thus, the amplitude response is,

$$|G\left(e^{j\omega}\right)| = |1 - H_R\left(\omega + \pi\right)| \tag{6}$$



(a) Amplitude response of $|H\left(e^{-(j\omega+\pi)}\right)|$ (b) Amplitude response of $|G\left(e^{j\omega}\right)|$

Figure 2: Plots

(P. P. Vaidyanathan, 3.13)

(Part a)

For real θ it is clear that $e^{\pm\theta}$ is real. The geometric mean of e^{θ} and $e^{-\theta}$ is 1. With the property that arithmetic mean is equal to or greater than the geometric mean, we have

$$x = \frac{e^{\theta} + e^{-\theta}}{2} \geq 1 \tag{7}$$

Now we assume θ is complex. Let $\theta = a + ib$. Thus we have,

$$\begin{aligned}\frac{e^{\theta} + e^{-\theta}}{2} &= \frac{e^{a+ib} + e^{-a-ib}}{2} \\ &= \frac{e^{a}\left(\cos b + i\sin b\right) + e^{-a}\left(\cos b - i\sin b\right)}{2} \\ &= \frac{\cos b \times \left(e^{a} + e^{-a}\right) + i\sin b \times \left(e^{a} - e^{-a}\right)}{2}\end{aligned} \tag{8}$$

For the above equation to have a real value either $b = 0$ or $(e^{a} - e^{-a}) = 0$. Clearly the former is not true, then θ will be real. Thus, taking the latter as true we have

$$\begin{aligned}(e^{a} - e^{-a}) &= 0 \\ \text{or } a &= 0.\end{aligned} \tag{9}$$

Thus, $\theta = ib$. To make x have a value in $[-1, 1]$ $a = 0$ and $b = j\omega$. With $\theta = j\omega$ we have,

$$\begin{aligned}x &= \frac{e^{j\omega} + e^{-j\omega}}{2} \\ &= \cos\omega\end{aligned}$$

Thus, $-1 \leq x \leq 1$.

(Part b)

$$\begin{aligned}\text{R.H.S} &= \cosh(N\theta)\cosh\theta \pm \sinh(N\theta)\sinh\theta \\ &= \frac{\left(e^{N\theta} + e^{-N\theta}\right)\left(e^{\theta} + e^{-\theta}\right) \pm \left(e^{N\theta} - e^{-N\theta}\right)\left(e^{\theta} - e^{-\theta}\right)}{4}\end{aligned} \tag{10}$$

Taking just the addition $'+'$ first, (subtraction can be also check similarly)

$$\begin{aligned}\text{R.H.S} &= \frac{e^{(N+1)\theta} + e^{\theta(1-N)} + e^{(N-1)\theta} + e^{-\theta(N+1)} + e^{(N+1)\theta} - e^{\theta(1-N)} - e^{(N-1)\theta} + e^{-\theta(N+1)}}{4} \\ &= \frac{2\left(e^{(N+1)\theta} + e^{-\theta(N+1)}\right)}{4} \\ &= \frac{e^{(N+1)\theta} + e^{-\theta(N+1)}}{2} \\ &= \cosh\left(\left(N+1\right)\theta\right) \\ &= \text{L.H.S.}\end{aligned} \tag{11}$$

The second part of this question is as follows,
We have,

$$\begin{aligned} C_N(x) &= \cosh(N\theta) \\ &= \frac{\left(e^{N\theta} + e^{-N\theta}\right)}{2} \end{aligned} \tag{12}$$

Thus,

$$\begin{aligned} 2xC_N(x) - C_{N-1}(x) &= 2x\cosh(N\theta) - \cosh((N-1)\theta) \\ &= 2x\frac{e^{N\theta} + e^{-\theta N}}{2} - \frac{e^{(N-1)\theta} + e^{-\theta(N-1)}}{2} \\ &= 2\left(\frac{e^{\theta} + e^{-\theta}}{2}\right)\frac{e^{N\theta} + e^{-\theta N}}{2} - \frac{e^{(N-1)\theta} + e^{-\theta(N-1)}}{2} \\ &= \frac{e^{\theta(N+1)} + e^{\theta(N-1)} + e^{\theta(1-N)} + e^{-\theta(1+N)} - e^{(N-1)\theta} - e^{-\theta(N-1)}}{2} \\ &= \frac{e^{\theta(N+1)} + e^{-\theta(N+1)}}{2} \\ &= \cosh((N+1)\theta) \\ &= C_{N+1}(x). \end{aligned} \tag{13}$$

Thus, we have proved

$$C_{N+1}(x) = 2xC_N(x) - C_{N-1}(x) \tag{14}$$

(Part b)

We have,

$$C_0(x) = 1 \tag{15}$$

$$C_1(x) = x \tag{16}$$

$$\begin{aligned} C_2(x) &= 2xC_1(x) - C_0(x) \\ &= 2x^2 - 1 \end{aligned} \tag{17}$$

$$\begin{aligned} C_3(x) &= 2xC_2(x) - C_1(x) \\ &= 2x(2x^2 - 1) - x \\ &= 4x^3 - 3x \end{aligned} \tag{18}$$

$$\begin{aligned} C_4(x) &= 2xC_3(x) - C_2(x) \\ &= 2x(4x^3 - 3x) - 2x^2 + 1 \\ &= 8x^4 - 8x^2 + 1 \end{aligned} \tag{19}$$

$$C_5(x) = 2xC_4(x) - C_3(x) \tag{20}$$

$$= 2x(8x^4 - 8x^2 + 1) - 4x^3 + 3x \tag{21}$$

$$= 16x^5 - 20x^3 + 5x. \tag{22}$$

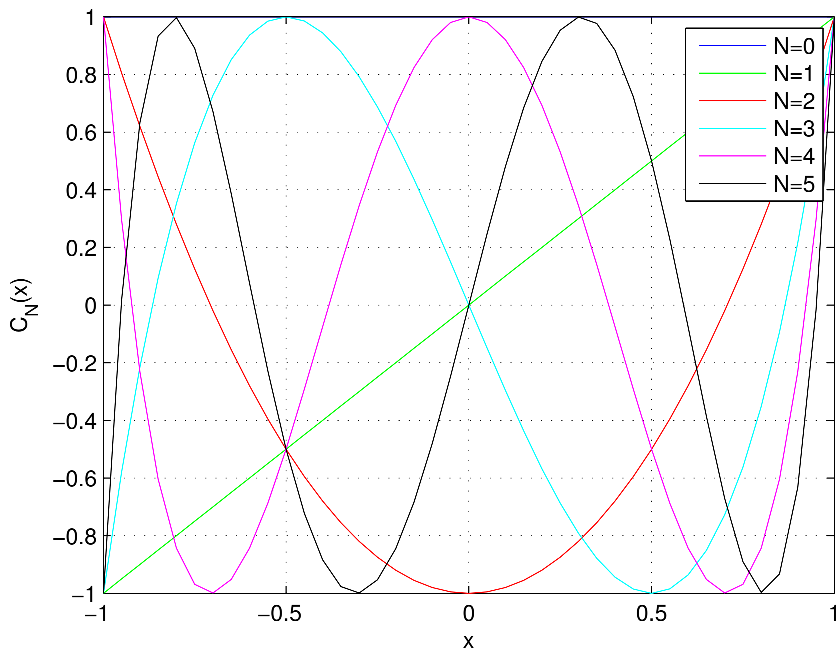


Figure 3: Plots of $C_N(x)$

(Part d)

$$C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x) \tag{23}$$

For,

$$C_0(x) = 1 \text{ [the required condition is true]}$$
$$C_1(x) = x \text{ [the required condition is true]}$$

We also assume the required condition is true for $0, 1, 2, ..., N-1$.

Case 1: When N is even.

$C_{N-1}(x)$ has odd powers, so $xC_{N-1}(x)$ has even powers. $C_{N-2}(x)$ also has even powers. Thus, C_N also has even powers.

Case 2: When N is odd.

$C_{N-1}(x)$ has even powers, so $xC_{N-1}(x)$ has odd powers. $C_{N-2}(x)$ also has odd powers. Thus, C_N also has odd powers.

Thus, our required condition is true for any N.

$$\begin{aligned} C_N(1) &= \cosh(N \cosh^{-1} 1) \\ &= \cosh(N \times 0) \\ &= 1. \end{aligned} \tag{24}$$

We have,

$$\begin{aligned} C_N(x) &= 2xC_{N-1}(x) - C_{N-2}(x) \\ &= 2^2x^2C_{N-2}(x) - 4xC_{N-3} - C_{N-4}(x) \end{aligned}$$

The first term R.H.S has the form of 2^k for the term $C_{N-k}(x)$. The first term is $C_{N-1}(x)$ for the polynomial $C_N(x)$ which implies that the corresponding coefficient is 2^{N-1}.

(Part e)

We have,

$$\begin{aligned} C_N(x) &= 0 \\ \text{or, } \cosh(N\theta) &= 0 \end{aligned}$$

This means,

$$\begin{aligned} N\theta &= jk\pi \\ \text{or, } \theta &= \frac{jk\pi}{N} \end{aligned}$$

Now,

$$\cosh(N\theta) = \cosh(N \cosh^{-1} x) = 0$$

Thus,

$$\begin{aligned} \cosh^{-1} x &= j\frac{k\pi}{N} \\ \text{or, } x &= \cosh\left(j\frac{k\pi}{N}\right) \\ &= \cos\left(\frac{k\pi}{N}\right) \end{aligned}$$

which proves $-1 \leq x \leq 1$.

(P. P. Vaidyanathan, 4.7)
For $0 \le k \le M-1$ the two sets are,

$$S = \left\{W^0, W^1, ..., W^{M-1}\right\} \text{ and } S_L = \left\{W^0, W^L, ..., W^{L(M-1)}\right\}$$

Necessary condition: Let us take k_1 and k_2 such that $0 \le k_1 < k_2 \le M-1$. Thus, $S = S_L$ for some k_1 and k_2 such that $0 \le k_1 < k_2 \le M-1$ with the condition

$$k_1 L \text{ mod } M = k_2 L \text{ mod } M$$
$$\text{or, } (k_1 - k_2)L \text{ mod } M = 0.$$

Since, $k_2 - k_1 \neq 0$ and $k_1 - k_2 \le M-1$, thus, M does not divide $k_2 - k_1$. Therefore, $(k_1 - k_2)L$ mod N = 0 holds true only for some factor (not equal to 1) of M divides L, which implies g.c.d$(M, L) = 1$. Thus, M and L are relatively prime.

Sufficient condition:

Let g.c.d$(M, L) = 1$. Since $0 \le k_1 < k_2 \le M-1$ for all $k_2 \neq k_1$, M does not divide $(k_2 - k_1)$. Which means, M also does not divide $(k_2 - k_1) \times L$ as g.c.d$(M, L) = 1$. Therefore,

$$(k_1 - k_2)L \text{ mod } M \neq 0.$$

Which means,

$$k_1 L \text{ mod } M \neq k_2 L \text{ mod } M$$

Hence, the elements kL mod M are all unique for $0 \le k \le M-1$. Therefore the set $S = S_L$.

(P. P. Vaidyanathan, 4.8)

(Part a)

We are given for figure (a)

$$y_1(n) = \begin{cases} x\left(\frac{Mn}{L}\right) & \text{where, } n = \text{multiple of } L. \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

For figure (b) let us consider $x_2(n)$ to be the function after the upsampler L. Thus,

$$x_2(n) = \begin{cases} x\left(\frac{n}{L}\right) & \text{where, } n = \text{multiple of } L. \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Finally we obtain

$$y_2(n) = \begin{cases} x\left(\frac{Mn}{L}\right) & \text{where, } Mn = \text{multiple of } L. \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

(Part a)

We apply z-transform on both $y_1(n)$ and $y_2(n)$,

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{L}{M}} W^k\right) \tag{28}$$

and,

$$Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{L}{M}} W^{kL}\right) \tag{29}$$

The difference between $Y_1(z)$ and $Y_2(z)$ is clearly in the powers of W. The first one is W^k and the second one is W^{kL}. In order for them to be equal L and M has to be relatively prime. The explanation is already given in problem 4.7.

(P. P. Vaidyanathan, 4.10)
We have,

$$x(n) = x(n+N) \tag{30}$$

$y(n)$ is a $M-$fold decimated version of $x(n)$. Thus,

$$\begin{aligned} y(n) &= x(Mn) && (31)\\ &= x(Mn+N) && (32) \end{aligned}$$

Now,

$$y(n+L) = x(Mn + ML - kN) \tag{33}$$

Here k is an integer, which means we are taking the k^{th} period of the function. In order to show, $y(n+L) = y(n)$, it is good enough to show $y(n+L) = x(Mn)$. Thus, this condition forces us to make

$$ML - kN = 0 \tag{34}$$

which is,

$$k = \frac{ML}{N} \tag{35}$$

Thus, we can easily choose a integer k for which $L < \infty$. We also have,

$$L = \frac{kN}{M} \tag{36}$$

In order to find the smallest L, we have to find smallest kN such that kN is divisible by M. kN is trivially divisible by N. Therefore the smallest kN is

$$\begin{aligned} kN &= \text{LCM}\,(M,N) \\ \implies k &= \frac{\text{LCM}\,(M,N)}{N} \\ \implies L &= \frac{kN}{M} = \frac{\text{LCM}\,(M,N)}{M}. \end{aligned}$$

(P. P. Vaidyanathan, 4.15)
The filter $H(z)$ can be decomposed into polyphase form as

$$H(z) = R_0\left(z^3\right) + z^{-1}R_1\left(z^3\right) + z^{-2}R_2\left(z^3\right).$$

The polyphase components $R_0(z)$, $R_1(z)$ and $R_2(z)$ can be further decomposed as

$$R_i(z) = R_{i0}\left(z^4\right) + z^{-1}R_{i1}\left(z^4\right) + z^{-2}R_{i2}\left(z^4\right) + z^{-3}R_{i3}\left(z^4\right), \quad i = 0, 1, 2.$$

Using the above polyphase decompositions, the given fractional decimation filter can be efficiently implemented as

(Part a) If the filter is implemented directly, the sample rate of the signals $x_1[n]$, $x_2[n]$ at the input and output of $H(z)$ is $L\times100$ KHz= 300 KHz. Therefore, the implementation of $H(z)$ must perform 60 multiplications per $\frac{1}{3}\times10^{-5}$ seconds. If the multiplications are performed parallely, each multiplier has $\frac{1}{3}\times10^{-5}\approx 3.33\,\mu s$. If the multiplications are performed serially, each multiplier has $\frac{1}{60}\times\frac{1}{3}\times10^{-5}\approx 55.56\,ns$.

(Part b) If the filter is implemented in efficient way using polyphase decomposition, sample rates at the input and output of the filters $R_{ij}(z)$ is $\frac{1}{M}\times 100$ KHz= 25 KHz. If the multiplications are performed parallely, each multiplier has $\frac{1}{25}\times10^{-3} = 40\,\mu s$.

(Part c) In the polyphase decomposition, the 60 coefficients of $H(z)$ are split across various polyphase components $R_{i,j}(z)$. Since, each $R_{ij}(z)$ operates at 25 KHz, the total number of multiplications performed per second $= 60\times25\times10^3 = 1.5\times10^6$.

If l_{ij} are the number of coefficients in $R_{ij}(z)$, then the number of additions required for one output sample is $l_{ij}-1$. We also know that $\sum_{i=0}^{2}\sum_{j=0}^{3} l_{ij} = 60$. We also perform 9 additions/sample at the output of $R_{ij}(z)$s. Therefore, the total number of additions performed per second $= 25\times10^3\times\left(\sum_{i=0}^{2}\sum_{j=0}^{3}(l_{ij}-1)+9\right) = 25\times10^3\times(60-12+9) = 1.425\times10^6$.

The additions after upsampling by 3 can be avoided because only one out of the three samples to be added will be non-zero. Therefore, these additions are not counted.

(P. P. Vaidyanathan, 4.16)
The given statement is not true.
We justify it with a counter example.
We have,

$$g(n) = h(2n) \tag{37}$$

Choose a all pass $G(z)$

$$G(z) = g_0 + g_1 z^{-1} + ...$$

with

$$\left|G\left(e^{j\omega}\right)\right| = 1.$$

In order to satisfy our given equation we can choose $H(z)$ as follows,

$$\begin{aligned} H(z) &= g_0 + 0.z^{-1} + g_1 z^{-2} + 0.z^{-3} + ... \\ &= G\left(z^2\right) \\ \implies H\left(e^{j\omega}\right) &= G\left(e^{2j\omega}\right) \\ \implies \left|H\left(e^{j\omega}\right)\right| &= \left|G\left(e^{2j\omega}\right)\right| = 1. \end{aligned}$$

Therefore, $H(z)$ is an all pass filter.

We have shown a counter example where $H(z)$ is an all pass filter and $g(n) = h(2n)$ is also an all pass filter and $H(z)$ is not an impluse function.

(P. P. Vaidyanathan, 4.21)

$$\begin{aligned} H_0(z) &= 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4} \\ &= \left(1 + 4z^{-2} + z^{-4}\right) + z^{-1}\left(2 + 2z^{-2}\right) \\ H_1(z) = H_0(-z) &= \left(1 + 4z^{-2} + z^{-4}\right) - z^{-1}\left(2 + 2z^{-2}\right) \end{aligned}$$

The polyphase components are

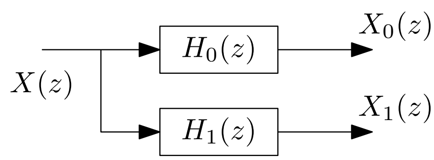
$$\begin{aligned} E_0\left(z^2\right) &= 1 + 4z^{-2} + z^{-4} \\ E_0\left(z^2\right) &= 2 + 2z^{-2}. \end{aligned}$$

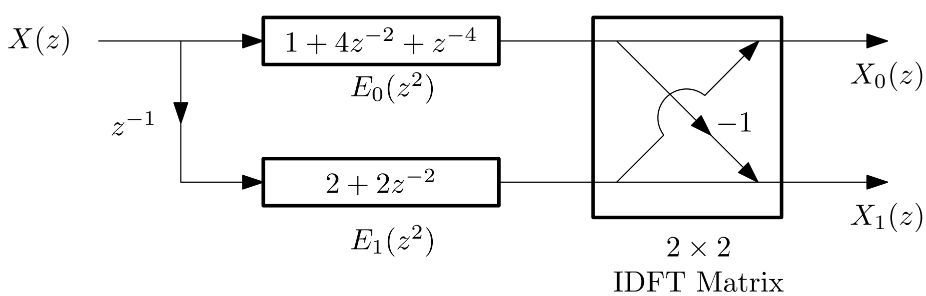
$$\begin{aligned} X_0(z) = H_0(z)\,X(z) &= \left[\left(1 + 4z^{-2} + z^{-4}\right) + z^{-1}\left(2 + 2z^{-2}\right)\right] X(z) \\ X_1(z) = H_1(z)\,X(z) &= \left[\left(1 + 4z^{-2} + z^{-4}\right) - z^{-1}\left(2 + 2z^{-2}\right)\right] X(z) \end{aligned}$$

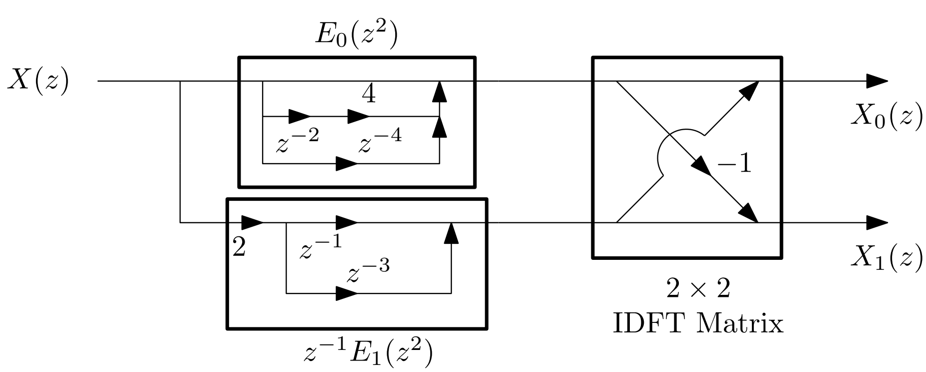
In matrix form,

$$\begin{bmatrix} X_0(z) \\ X_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{2\times 2\,\text{IDFT Matrix}} \begin{bmatrix} \left(1 + 4z^{-2} + z^{-4}\right) \\ z^{-1}\left(2 + 2z^{-2}\right) \end{bmatrix} X(z)$$

The implementation is given in the figure below.







(P. P. Vaidyanathan, 4.27)

$$H_0(z) = \sum_{i=0}^{N} h_0(i) z^{-i}$$

$$H_k(z) = H\left(zW^k\right) = \sum_{i=0}^{N} h_0(i) \left(zW^k\right)^{-i} = \sum_{i=0}^{N} h_0(i) W^{-ik} z^{-i}$$

$$\Longrightarrow h_k(i) = h_0(i) W^{-ik}, \quad i = 0, \cdots N; \quad k = 0, 1, 2, 3, 4.$$

(Part a) If $h_1(1) = h_0(1) W^{-1}$. Since $h_0(1)$ is real and $W^{-1} = e^{-\frac{j2\pi}{5}}$ is complex, $h_1(1)$ is complex. Therefore, $h_k(n), 1 \le k \le 4$ are not all real for all n.

(Part b)

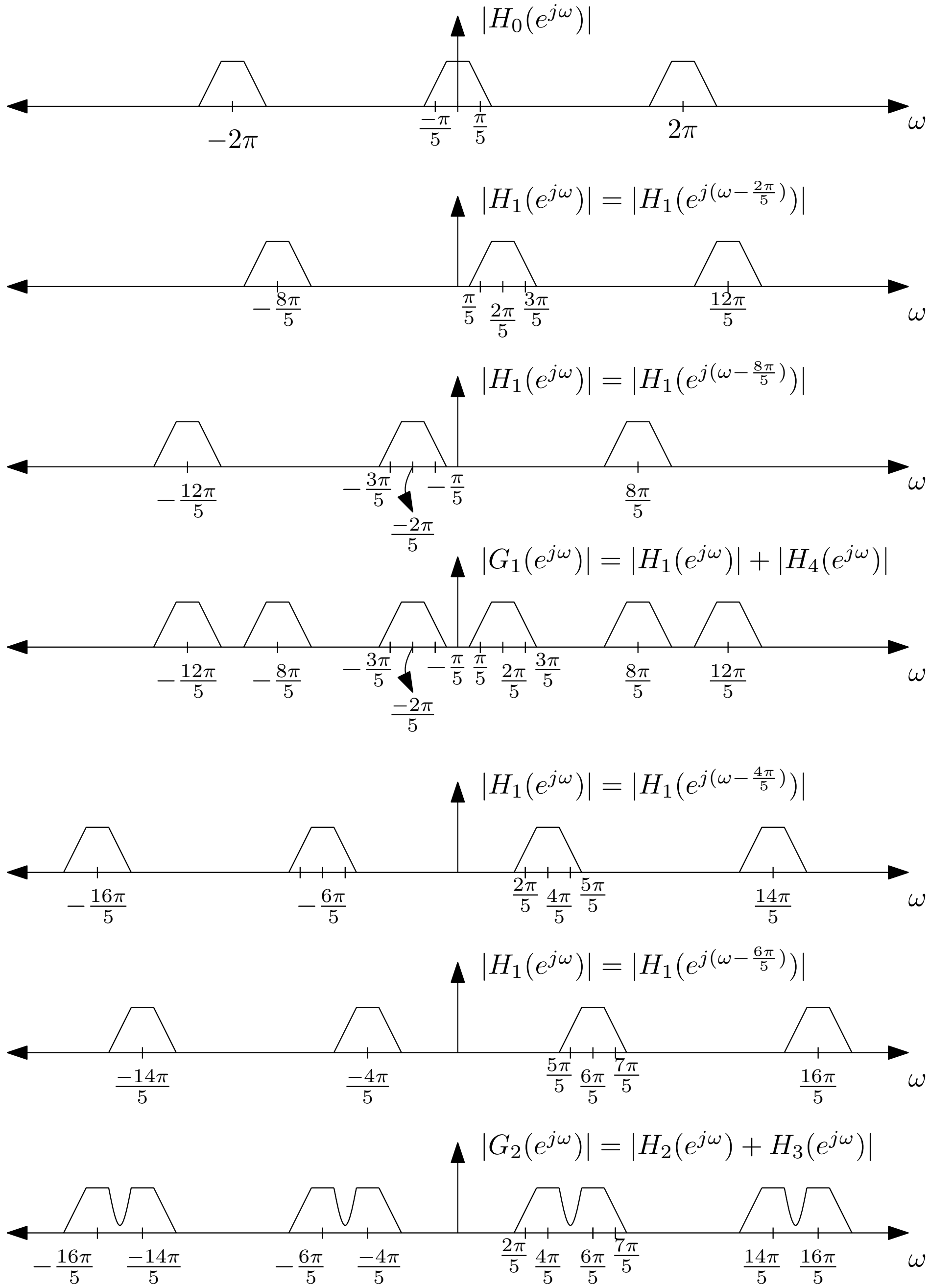
$$\begin{aligned}
G_1(z) &= H_1(z) + H_4(z) \\
&= \sum_{i=0}^{N} h_0(i) W^{-i} z^{-i} + \sum_{i=0}^{N} h_0(i) W^{-4i} z^{-i} \\
&= \sum_{i=0}^{N} h_0(i) \left(W^{-i} + W^{-4i}\right) z^{-i} \\
&= \sum_{i=0}^{N} h_0(i) \left(W^{-i} + W^{i}\right) z^{-i} \quad \left(W^5 = 1 \Longrightarrow W^{-4i} = W^{5i-4i}\right) \\
&= \sum_{i=0}^{N} h_0(i) \left(e^{-j\frac{2i\pi}{5}} + e^{j\frac{2i\pi}{5}}\right) z^{-i} \\
G_1(z) &= \sum_{i=0}^{N} h_0(i) \cos\left(\frac{2i\pi}{5}\right) z^{-i} \\
\Longrightarrow g_1(n) &= h_0(n) \cos\left(\frac{2\pi}{5} n\right)
\end{aligned}$$

$$\begin{aligned}
G_2(z) &= H_2(z) + H_3(z) \\
&= \sum_{i=0}^{N} h_0(i) W^{-2i} z^{-i} + \sum_{i=0}^{N} h_0(i) W^{-3i} z^{-i} \\
&= \sum_{i=0}^{N} h_0(i) \left(W^{-2i} + W^{-3i}\right) z^{-i} \\
&= \sum_{i=0}^{N} h_0(i) \left(W^{-2i} + W^{2i}\right) z^{-i} \quad \left(W^5 = 1 \Longrightarrow W^{-3i} = W^{5i-4i}\right) \\
&= \sum_{i=0}^{N} h_0(i) \left(e^{-j\frac{4i\pi}{5}} + e^{j\frac{4i\pi}{5}}\right) z^{-i} \\
G_2(z) &= \sum_{i=0}^{N} h_0(i) \cos\left(\frac{4i\pi}{5}\right) z^{-i} \\
\Longrightarrow g_2(n) &= h_0(n) \cos\left(\frac{4\pi}{5} n\right)
\end{aligned}$$

Therefore, $g_1(n)$ and $g_2(n)$ are all real.

(Part c)

$\left|G_2\left(e^{j\omega}\right)\right|$ need not necessarily look 'good' in the pass band. However, if the phase responses of $H_2\left(e^{j\omega}\right)$ and $H_3\left(e^{j\omega}\right)$ are equal in the overlapping regions, the magnitude response $\left|G_2\left(e^{j\omega}\right)\right|$ will be constant in the pass band.



(P. P. Vaidyanathan, 4.28)
Writing the polyphase decomposition of $H_0(z)$:

$$\begin{aligned} H_0(z) &= E_0\left(z^2\right) + z^{-1}E_1\left(z^2\right) \\ H_1(z) = H_0(-z) &= E_0\left(z^2\right) - z^{-1}E_1\left(z^2\right). \end{aligned}$$

From the figure, the output signal is

$$\begin{aligned} \hat{X}(z) &= \left(F_0(z)H_0(z) + F_1(z)H_1(z)\right)X(z) \\ \implies \frac{\hat{X}(z)}{X(z)} &= E_0\left(z^2\right)\left(F_0(z) + F_1(z)\right) + z^{-1}E_1\left(z^2\right)\left(F_0(z) - F_1(z)\right). \end{aligned}$$

(Part a)

$$\begin{aligned} H_0(z) &= 1 + 3z^{-1} + 0.5z^{-2} + z^{-3} \\ \implies E_0\left(z^2\right) &= 1 + 0.5z^{-2} \\ z^{-1}E_1\left(z^2\right) &= 3z^{-1} + z^{-3} \end{aligned}$$

$$\frac{\hat{X}(z)}{X(z)} = \left(1 + 0.5z^{-2}\right)\left(F_0(z) + F_1(z)\right) + \left(3z^{-1} + z^{-3}\right)\left(F_0(z) - F_1(z)\right)$$

We can choose $F_0(z) = F_1(z) = \frac{1}{2(1+0.5z^{-2})} \implies \frac{\hat{X}(z)}{X(z)} = 1$. Note that $F_0(z)$ and $F_1(z)$ are causal and stable because the poles are located at $\pm j\frac{1}{\sqrt{2}}$ (inside unit circle).

(Part b)

$$\begin{aligned} H_0(z) &= 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \\ \implies E_0\left(z^2\right) &= 1 + 3z^{-2} + z^{-4} \\ z^{-1}E_1\left(z^2\right) &= 2\left(z^{-1} + z^{-3}\right) \end{aligned}$$

$$\begin{aligned} \frac{\hat{X}(z)}{X(z)} &= \left(1 + 3z^{-2} + z^{-4}\right)\left(F_0(z) + F_1(z)\right) + 2\left(z^{-1} + z^{-3}\right)\left(F_0(z) - F_1(z)\right) \\ &= \left(\left(1 + z^{-2}\right)^2 + z^{-2}\right)\left(F_0(z) + F_1(z)\right) + 2z^{-1}\left(1 + z^{-2}\right)\left(F_0(z) - F_1(z)\right) \end{aligned}$$

Choose

$$\begin{aligned} F_0(z) + F_1(z) &= z^{-1} \\ F_0(z) - F_1(z) &= -\frac{1}{2}\left(1 + z^{-2}\right) \end{aligned}$$

$$\begin{aligned} \implies \frac{\hat{X}(z)}{X(z)} &= z^{-1}\left(1 + z^{-2}\right)^2 + z^{-3} - z^{-1}\left(1 + z^{-2}\right)^2 \\ &= z^{-3} \end{aligned}$$

Therefore, perfect reconstruction is possible with following causal FIR filters:

$$\begin{aligned} F_0(z) &= \frac{1}{4}\left(-1 + z^{-2}\right) \\ F_1(z) &= \frac{1}{4}\left(1 + 3z^{-2}\right). \end{aligned}$$