E9-252: Mathematical Methods and Techniques in Signal Processing
Homework 4 Solutions

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Problem 4.4

$$\hat{H}(z) = \frac{b(0) + b(1)z^{-1} + b(2)z^{-2}}{1 + a(1)z^{-1} + a(2)z^{-2}}$$

Part a:
$h(0) = -1$; $h(1) = 2$; $h(2) = 3$; $h(3) = 2$; $h(4) = 1$
Using Pade's method, we have

$$\begin{bmatrix} h(0) & 0 & 0 \\ h(1) & h(0) & 0 \\ h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} b(0) \\ b(1) \\ b(2) \end{bmatrix}$$

$$\begin{bmatrix} h(3) & h(2) & h(1) \\ h(4) & h(3) & h(2) \end{bmatrix} \begin{bmatrix} 1 \\ a(1) \\ a(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving the above equations gives us

$$a(1) = -\frac{4}{5};\ a(2) = \frac{1}{5};\ b(0) = -1;\ b(1) = \frac{14}{5};\ b(0) = \frac{6}{5};$$

$$\implies \hat{H}(z) = \frac{-1 + \frac{14}{5}z^{-1} + \frac{6}{5}z^{-2}}{1 - \frac{4}{5}z^{-1} + \frac{1}{5}z^{-2}}.$$

Part b:
If there is noise in the observed signal, we cannot determine the order of the system. Assuming that there is no noise, Pade's method should give exact model if the order is correct. From the model obtained using Pade's method,

$$\hat{h}(5) = \frac{2}{5} \neq h(5).$$

Therefore, the hypothesis about the order is not correct.

Problem 4.11

$$H(z) = \frac{B_q(z)}{A_p(z)} = \frac{b(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-2k}}$$

For Prony's method, we define the error as

$$E(z) = X(z) A_p(z) - B_q(z)$$

$$\implies e(n) = x(n) + \sum_{k=1}^{p} a_p(k) x(n-2k) - b_0 \delta(n).$$

We minimize the squared error given by

$$\mathcal{E}_{p,q} = \sum_{k=q+1}^{\infty} |e(k)|^2$$

$$\frac{\partial \mathcal{E}_{p.q}}{\partial a_p^*(l)} = 0, \quad l = 1, 2, \cdots, p$$

$$\implies \sum_{k=1}^{\infty} \left[x(k) + \sum_{g=1}^{k} a_p(g) x(k-2g) \right] x^*(k-2l) = 0$$

$$\implies \sum_{g=1}^{k} a_p(q) r_x(2l, 2k) = -r_x(2l, 0), \quad l = 1, 2, \cdots, p$$

where $r_x(k,l) = \sum_{n=1}^{\infty} x(n-l) x^*(n-k)$. Writing the above equations in matrix form, we have

$$\underbrace{\begin{bmatrix} r_x(2,2) & r_x(2,4) & \cdots & r_x(2,2p) \\ r_x(4,2) & r_x(4,4) & \cdots & r_x(4,2p) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(2p,2) & r_x(2p,4) & \cdots & r_x(2p,2p) \end{bmatrix}}_{R_x} \underbrace{\begin{bmatrix} a_p(1) \\ a_p(2) \\ \vdots \\ a_p(p) \end{bmatrix}}_{\underline{a}_p} = - \underbrace{\begin{bmatrix} r_x(2,0) \\ r_x(4,0) \\ \vdots \\ r_x(2p,0) \end{bmatrix}}_{\underline{r}_x}$$

$$\implies R_x \underline{a}_p = -\underline{r}_x.$$

We can solve the above equations by inverting R_x if R_x is full rank matrix. A pseudo-inverse can be used if R_x is it is not a full rank matrix.

Problem 4.15

$$H(z) = \frac{b(0)}{1 - a(1) z^{-1}}$$
$$\Longrightarrow \; h(n) = b(0) (a(1))^n u(n).$$

The squared error is

$$\begin{aligned}
\mathcal{E} &= \sum_{n=0}^{N-1} [x(n) - h(n)]^2 \\
&= \sum_{n=0}^{N-1} [x(n) - b(0)(a(1))^n]^2
\end{aligned}$$

To minimize $\mathcal{E}$,

$$\frac{\partial \mathcal{E}}{\partial b(0)} = 0$$

$$\Longrightarrow \; -\sum_{n=0}^{N-1} 2 (a(1))^n [x(n) - b(0)(a(1))^n] = 0$$
$$\Longrightarrow \; b(0) \frac{(a(1))^{2N} - 1}{(a(1))^2 - 1} = \sum_{n=0}^{N-1} (a(1))^n x(n)$$
$$\Longrightarrow \; b(0) = \frac{(a(1))^2 - 1}{(a(1))^{2N} - 1} \sum_{n=0}^{N-1} (a(1))^n x(n).$$

This gives $b(0)$ as a function of $a(1)$.
Similarly,

$$\frac{\partial \mathcal{E}}{\partial a(1)} = 0$$
$$\Longrightarrow \; -\sum_{n=0}^{N-1} n b(0) (a(1))^{n-1} [x(n) - b(0)(a(1))^n] = 0$$
$$\Longrightarrow \; b(0) \sum_{n=0}^{N-1} n (a(1))^{n-1} x(n) - b(0) a(1) \sum_{n=0}^{N-1} n \left((a(1))^2\right)^{n-1} = 0 \tag{1}$$

We know that

$$\sum_{n=0}^{N-1} r^n = \frac{r^N - 1}{r - 1}$$
$$\Longrightarrow \; \sum_{n=0}^{N-1} n r^{n-1} = \frac{d}{dr} \sum_{n=0}^{N-1} r^n = \frac{N r^{N-1}}{r - 1} - \frac{r^N - 1}{(r-1)^2} = \frac{(N-1) r^N - N r^{N-1} + 1}{(r-1)^2}$$

Therefore, we can write (1) as

$$b(0) \sum_{n=0}^{N-1} n (a(1))^{n-1} x(n) - b(0) a(1) \frac{(N-1)(a(1))^{2N} - N (a(1))^{2N-2} + 1}{(a(1))^2 - 1} = 0$$

$$\Longrightarrow \; b(0) \left[\left((a(1))^2 - 1\right) \left(\sum_{n=0}^{N-1} n (a(1))^{n-1} x(n) \right) + (N-1)(a(1))^{2N+1} - N (a(1))^{2N+1} + 1 \right] = 0$$

Since $b\left(0\right)=0$ gives $H\left(z\right)=0$, $b\left(0\right)\neq 0$. Therefore, the above equation becomes

$$\left(\left(a\left(1\right)\right)^2-1\right)\left(\sum_{n=0}^{N-1} n\left(a\left(1\right)\right)^{n-1}x\left(n\right)\right)+\left(N-1\right)\left(a\left(1\right)\right)^{2N+1}-N\left(a\left(1\right)\right)^{2N+1}+1=0.$$

$a\left(1\right)$ can be obtained by solving the above polynomial of order $\left(2N+1\right)$. To make sure that the solution to the polynomial minimizes $\mathcal{E}$, we need to check if $\frac{\partial^2\mathcal{E}}{\partial a\left(1\right)^2}>0$.

Problem 2

Part 1:

Given

$$f(t) = \begin{cases} \pi - t, & 0 \le t \le \pi \\ & -\pi < t < 0 \end{cases}$$

and $f(t) = f(t + 2\pi)$.

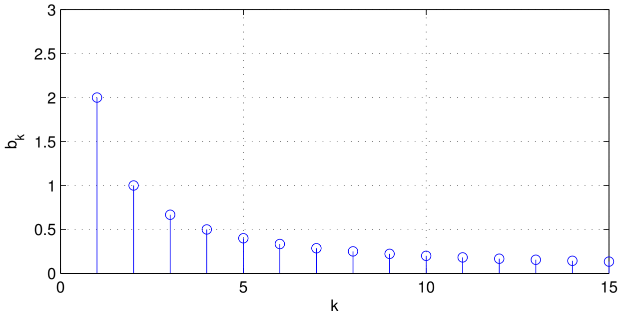
The Fourier series coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt = 0, \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\, dt = 0, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\, dt = \frac{2}{k}; \end{aligned}$$

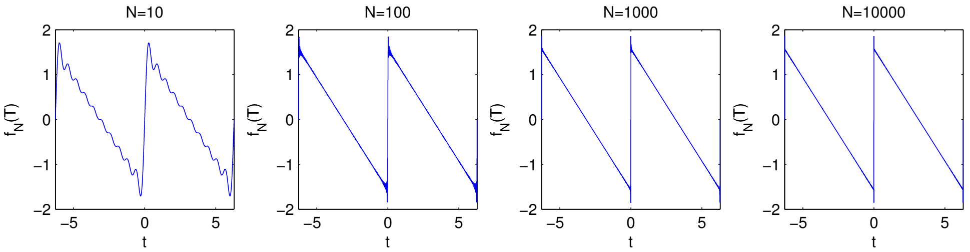
Therefore,

$$f(t) = \sum_{k=1}^{\infty} \frac{2}{k} \sin(kt).$$

The Fourier series coefficients are sketched below:



$f(t)$ approximated using upto N terms of the Fourier series is plotted below for different values of N:



Part 2:

$$g_N(x) = 2 \sum_{n=1}^{N} \frac{\sin(nx)}{n} - (\pi - x)$$

Using the result derived in the class,

$$g_N'(x) = 2 \left[\sum_{n=1}^{N} \cos(nx) + \frac{1}{2} \right] = \begin{cases} \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}, & x \neq 0 \\ 2N + 1, & x = 0 \end{cases}.$$

Therefore,

$$\begin{aligned} g_N'(x) &= 0 \\ \implies \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}} &= 0, \quad x \neq 0 \\ \implies \sin\left(N+\frac{1}{2}\right)x &= 0, \quad x \neq 0 \\ \implies x &= \frac{k\pi}{\left(N+\frac{1}{2}\right)}, \quad k \in \mathbb{Z}\setminus\{0\} \end{aligned}$$

Therefore, the smallest positive root of $g_N'(x)$ is $\theta_N = \frac{2\pi}{2N+1}$.
Following shows the behavior of $g_N(\theta_N)$ as a function of N.

N	10	50	100	500	1000	10^8
$g_N(\theta_N)$	0.56465848	0.56238411	0.56230737	0.56228249	0.56228171	0.56228145

We observe that

$$\lim_{N\to\infty} g_N(\theta_N) \approx 0.56228$$

Interpretation: This is the Gibb's phenomenon that is observed at the point of discontinuity ($x = 0$ i.e., $N \to \infty$, $\theta_N \to 0$). There is $\lim_{N\to\infty} g_N(\theta_N) = 0.5623$ shows that the Fourier approximation of the signal up to N terms results in an overshoot of the signal near θ_N. As $N \to \infty$, the overshoot appears near the point of discontinuity.