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Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

Instructor: Shayan Srinivasa Garani

Mid Term Exam#2, Fall 2017

Name and SR.No:

Instructions:

- You are allowed only 5 pages of written notes and a calculator for this exam. No wireless allowed.
- The time duration is 3 hrs.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: An analog signal $s(t)$ has a spectrum as shown in Figure 1. The maximum frequency in the signal as per the spectrum is $\omega_{\max}$ Hz. According to the sampling theorem, we must be able to reconstruct the signal by sampling at a rate $f_s > 2\omega_{\max}$ followed by an anti-aliasing filter. Is it possible to sample at less than the Nyquist rate and reconstruct the signal? Discuss the situation carefully. (15 pts.)
Hint: You do not need any compressive sampling or other recent sampling ideas except what has been discussed in the class.

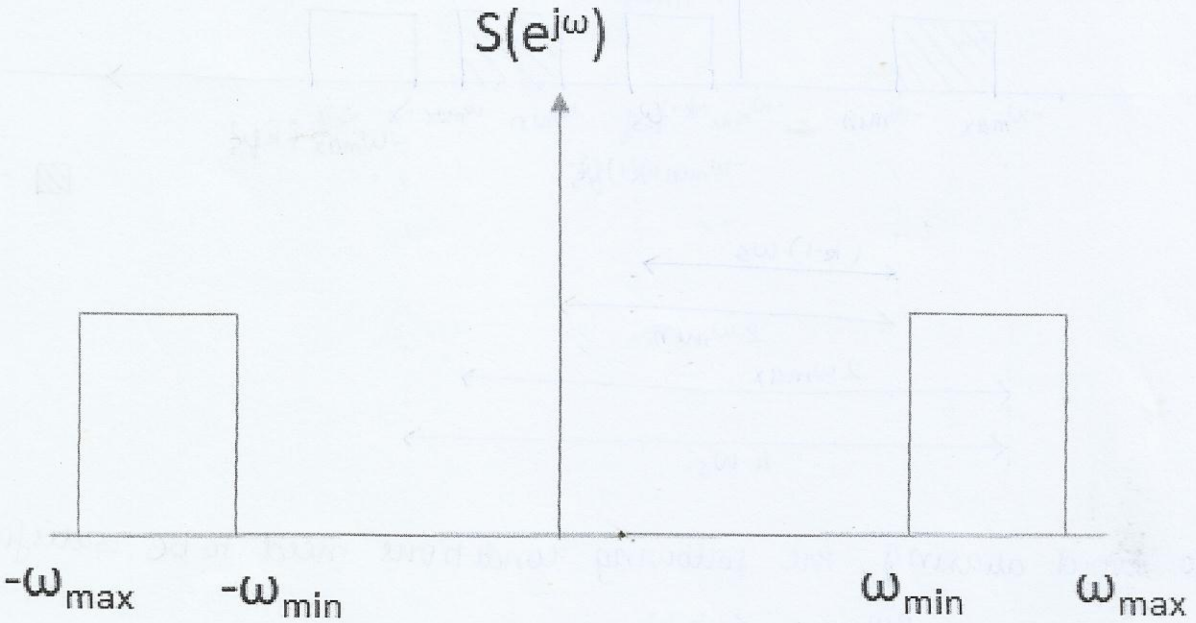


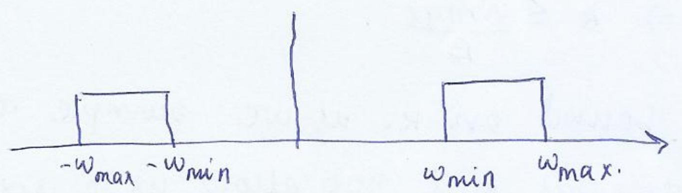
FIGURE 1. Spectrum of an analog signal.

Let f_s be the sampling frequency, then, the sampled signal spectrum is given by

$$S_s(f) = f_s \sum_{m=-\infty}^{\infty} s(f - m f_s)$$

$$S_s(e^{j\omega}) = f_s \sum_{m=-\infty}^{\infty} s(e^{j(\omega - 2\pi m f_s)}) \rightarrow ①$$

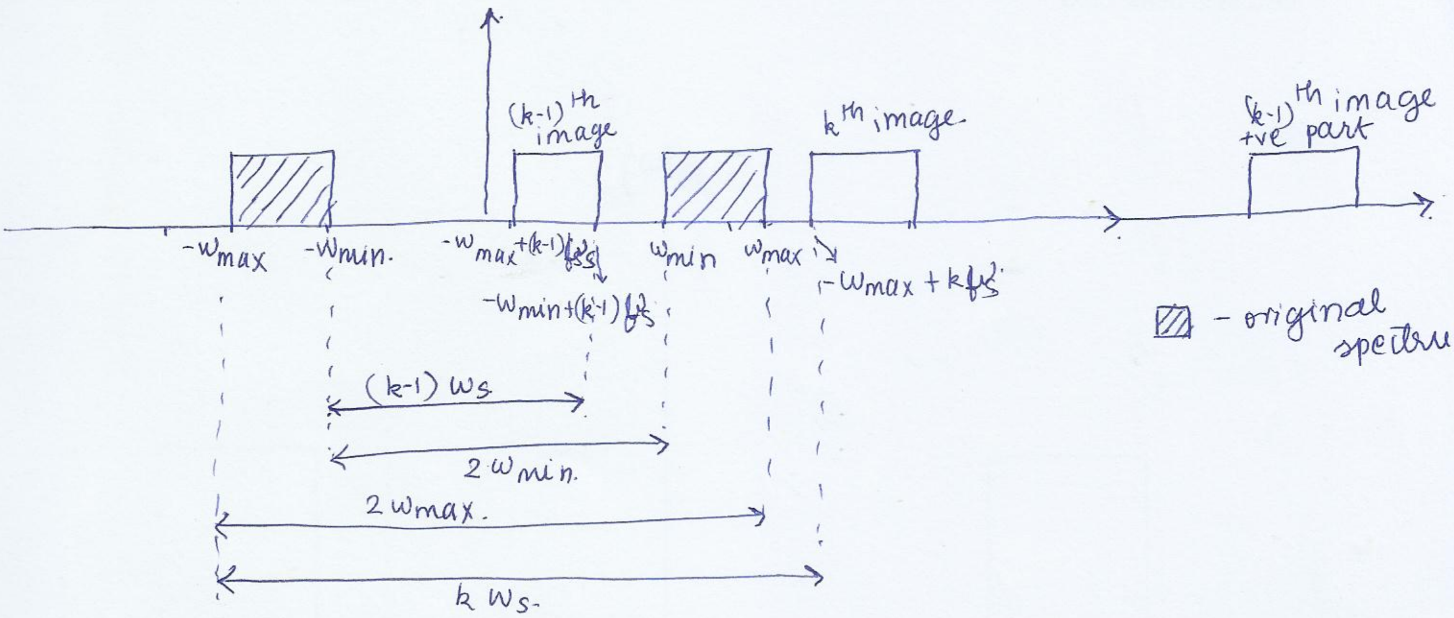
$s(e^{j\omega})$ is given by:



Now, the sampled signal is obtained by equation 1, which involves translating the spectrum and adding all those translations to obtain the spectrum of the sampled signal.

If there is any overlap of any of these translated spectrums (called images), there will be aliasing and hence we cannot reconstruct the signal. Let us now check under what conditions aliasing can be prevented.

Consider the $(k-1)^{th}$ and the k^{th} image, then, we have.



To avoid aliasing, the following conditions need to be satisfied.

$$2w_{max} \leq k w_s$$
$$(k-1) w_s \leq 2 w_{min}$$

$$\Rightarrow \frac{2w_{max}}{k} \leq w_s \leq \frac{2w_{min}}{k-1}$$

Now, $2w_{max} \leq k w_s \Rightarrow \frac{1}{w_s} \leq \frac{k}{2w_{max}}$.

$(k-1) w_s \leq 2 w_{min} \Rightarrow (k-1) w_s \leq 2 (w_{max} - B)$ $\quad (\because w_{max} - w_{min} = B)$

$$\therefore (k-1) \frac{2w_{max}}{k} \geq 2 (w_{max} - B)$$

$$2w_{max}k - 2w_{max} \geq 2w_{max}k - 2Bk.$$

$$2Bk \leq 2w_{max}. \quad \Rightarrow \quad k \leq \frac{w_{max}}{B}$$

Hence, there is an upper bound on k. if we sample at a frequency less than the nyquist rate. This will not allow us to reconstruct the signal.

PROBLEM 2: A student was performing measurements on a frequency reporting device. In an experimental report, the student mentioned that, while experimenting with pure sine waves, to account for frequency deviations of the device, he averaged the signal over a 0.5 ms time interval. He claimed that the device was producing frequencies accurately up to 0.1 KHz. Are the conclusions correct? Justify. (10 pts.)

$$\Delta t = 0.5 \times 10^{-3} s$$

$$\Delta f = 0.1 \times 10^{3} Hz$$

$$\Delta \omega = 2\pi \times 0.1 \times 10^{3} Hz$$

From time frequency uncertainity principle, we know that

$$\Delta \omega \, \Delta t \geq 2\pi \quad - \; 1$$

Substituting $\Delta \omega$ and Δt in 1, we get

$$\Delta \omega \, \Delta t = 2\pi \times 0.1 \times 10^{3} \times 0.5 \times 10^{-3} = 0.05 \times 2\pi \ngeq 2\pi \quad - \; 2$$

Therefore, from 1 and 2, we see that the student's claim is not correct.

Problem 3 : $f \subset [-2^{-p}, 2^{-p}]$. V_p is a space of all square integrable functions $f \in L^2(R)$
To examine if $\{V_p\}_{p \in Z}$ is an MRA.

Solution :

Nesting :

$$V_p = \left\{ f \in L^2(R) : \text{support}(f) \subset [-2^{-p}, 2^{-p}] \right\}$$

$$V_{p-1} = \left\{ f \in L^2(R) : \text{support}(f) \subset [-2^{-(p-1)}, 2^{-(p-1)}] \right\}$$

Now $[-2^{-p}, 2^{-p}] \subset [-2^{-(p-1)}, 2^{-(p-1)}]$ So it is a nested space

CLOSURE :

Also A) $\lim_{p \to \infty} V_p = \{0\}$.

B) $\lim_{p \to -\infty} V_p = L^2(\mathbb{R})$

SCALING : Consider $f \in V_p$ $[-2^{-(p+1)}, 2^{-(p+1)}]$

$\therefore$ support $f(2t) \subset [-2^{-(p+1)}, 2^{-(p+1)}] \subset [-2^{-(p+1)}, 2^{-(p+1)}]$

$f(2t) \in V_{p+1}$

Now $f(t) \in V_j \Rightarrow f(2^{-j} t) \in V_0$
i.e., support $[-2^{-j}, 2^{-j}]$ is stretched to $[-1, 1] \in V_0$

Shift Invariance

Consider $f \in V_0$, support $(f) \subset [-1, 1]$

Consider $k \neq 0$ Support $f(t-k) = [-1+k, 1+k]$

Clearly support $f(t-k) \not\subset [-1, 1]$

$\therefore f(t-k)$ is not necessarily in V_0

(FAILS)

BASIS

Since shift invariance is violated, $\{ \cancel{f}(t-k) \}_{k \in Z}$ will not form a basis

will not Constitute a

$\therefore \{ V_p \}_{p \in Z}$

MRA !

PROBLEM 4: Expand the signal $s(t) = t^2$ over the interval [-1,1] using Haar wavelets up to a resolution of 0.25. Sketch the wavelet expansion. What is the approximation error? (20 pts.)

The coefficients $a_{-1}^{(0)}, a_0^{(0)}, b_{-1}^{(0)}, b_0^{(0)}$ can be obtained by projecting $s(t)$ onto orthonormal basis:

$$a_{-1}^{(0)} = \langle \phi(t+1), s(t) \rangle = \int_{-1}^{0} t^2 dt = \frac{1}{3}$$

$$a_0^{(0)} = \int_0^1 t^2 dt = \frac{1}{3}$$

$$b_k^{(n)} = 2^{n/2} \cdot \left(\int_{\frac{k}{2^n}}^{\frac{k}{2^n}+\frac{1}{2^{n+1}}} t^2 dt - \int_{\frac{k}{2^n}+\frac{1}{2^{n+1}}}^{\frac{k+1}{2^n}} t^2 dt \right)$$

$$= \frac{2^{n/2}}{3} \left[\left(\frac{k}{2^n} + \frac{1}{2^{n+1}}\right)^3 - \left(\frac{k}{2^n}\right)^3 - \left(\frac{k+1}{2^n}\right)^3 + \left(\frac{k}{2^n} + \frac{1}{2^{n+1}}\right)^3 \right]$$

$$= \frac{2^{n/2}}{3} \left[2 \cdot \frac{(2k+1)^3}{(2^{n+1})^3} - \frac{k^3 + (k+1)^3}{(2^n)^3} \right]$$

$$= \frac{2^{n/2}}{3(2^{n+1})^3} \left[2 \cdot (2k+1)^3 - 2^3 (k^3 + (k+1)^3) \right]$$

$$\Rightarrow b_0^{(0)} = \frac{1}{3 \times 8} \left[2 \cdot 1 - 2^3 (0+1) \right] = \frac{-1}{4}.$$

$$b_0^{(1)} = \frac{2^{1/2}}{3(2^2)^3} \left[2 \cdot (1)^3 - 2^3 (0+1) \right] = \frac{\sqrt{2}}{3 \times 64} [2-8] = -\frac{\sqrt{2}}{32}.$$

$$b_1^{(1)} = \frac{2^{1/2}}{3(2^2)^3} \left[2 \cdot (3)^3 - 2^3 (1+8) \right] = \frac{\sqrt{2}}{3 \times 64} [54 - 72] = -\frac{3\sqrt{2}}{32}$$

$$b_{-1}^{(0)} = \frac{2^0}{3 \cdot 2^3} \left[2 \cdot (-2+1)^3 - 2^3 (-1+0) \right] = \frac{1}{24} [-2+8] = \frac{1}{4}.$$

$$b_{-1}^{(1)} = \frac{2^{1/2}}{3(2^2)^3} \left[2 (-2+1)^3 - 2^3 (-1+0) \right] = \frac{\sqrt{2}}{3 \times 64} [-2+8] = \frac{\sqrt{2}}{32}.$$

$$b_{-2}^{(1)} = \frac{2^{1/2}}{3(2^2)^3} \left[2 \cdot (-4+1)^3 - 2^3 (-8 \cancel{+} -1) \right] = \frac{\sqrt{2}}{3 \times 64} [-10] = -\frac{5\sqrt{2}}{96}.$$

$$\hat{f}(t) = a_0^{(0)} \phi(t) + a_{-1}^{(0)} \phi(t+1) + b_0^{(0)} \psi(t) + b_{-1}^{(0)} \psi(t+1) + b_0^{(1)} \psi(2t)$$
$$+ b_1^{(1)} \psi(2t-1) + b_{-1}^{(1)} \psi(2t+1) + b_{-2}^{(1)} \psi(2t+2)$$

$$\hat{y}(t) = \begin{cases} \frac{56-5\sqrt{2}}{96} & t \in [-1,-0.75] \cup [0.75,1] \\ \frac{56+5\sqrt{2}}{96} & t \in [-0.75,-0.5] \\ \frac{1}{12} + \frac{\sqrt{2}}{32} & t \in [-0.5,-0.25] \\ \frac{1}{12} - \frac{\sqrt{2}}{\sqrt{32}}. & t \in [-0.25, 0] \\ \frac{1}{12} - \frac{\sqrt{2}}{32}. & t \in [0, 0.25] \\ \frac{1}{12} + \frac{\sqrt{2}}{32} & t \in [0.25, 0.5] \\ \frac{56+5\sqrt{2}}{96} & t \in [0.5, 0.75] \\ \frac{56-5\sqrt{2}}{96} & t \in [0.75, 1] \end{cases}$$

Approximation error $= \int_{-1}^{1} |\hat{y}(t) - 8(t)|^2 dt$

$= 2\int_0^1 |\hat{f}(t) - 1(t)|^2 dt$

$$= 2\left[\left(\frac{56-5\sqrt{2}}{96}\right)^2 (0.25 + \frac{(0.25)^5}{5} - 2\left(\frac{56-5\sqrt{2}}{96}\right)(0.25)^3\right]$$

$$+ 2\left[\left(\frac{56+5\sqrt{2}}{96}\right)^2 (0.25) + \frac{(0.25)^5}{5} - 2\left(\frac{56+5\sqrt{2}}{96}\right)(0.25)^3\right]$$

$$+ 2\left[\left(\frac{1}{12}+\frac{\sqrt{2}}{32}\right)^2 (0.25) + \frac{(0.25)^5}{5} - 2\left(\frac{1}{12}+\frac{\sqrt{2}}{32}\right)(0.25)^3\right]$$

$$+ 2\left[\left(\frac{1}{12}-\frac{\sqrt{2}}{32}\right)^2 (0.25) + \frac{(0.25)^5}{5} - 2\left(\frac{1}{12}-\frac{\sqrt{2}}{32}\right)(0.25)^3\right]$$

PROBLEM 5: You are given a digital low pass FIR filter $H(z)$ of order N and no other filter. You are also given a sampled speech signal $x[n]$ from an analog signal of duration 2 minutes sampled at 16KHz using a 12 bit ADC. You are required to realize a speech compression engine.

(1) What is the original bit rate and the signal length in bits? (5 pts.)

(2) Suppose we decide to have J stages over dyadic subband coding as shown in Figure 2(A). Show the equivalent analysis bank of filters i.e., just an equivalent filter and a downsampler on each branch as in Figure 2(B) . What is the bit rate without any compression at each branch? You can assume that linear convolution is done. (7 pts.)

(3) Suppose we need a target compression rate of 1:R, propose a scheme as to how you would reduce the bit rate using just quantizers. You may use a uniform or non-uniform quantizer as appropriate. Justify your choice. (10 pts.)

(4) Sketch the synthesis stage, and show all the equivalent synthesis filters in each branch along with the necessary expanders similar to sub-part (2) above. (8 pts.)

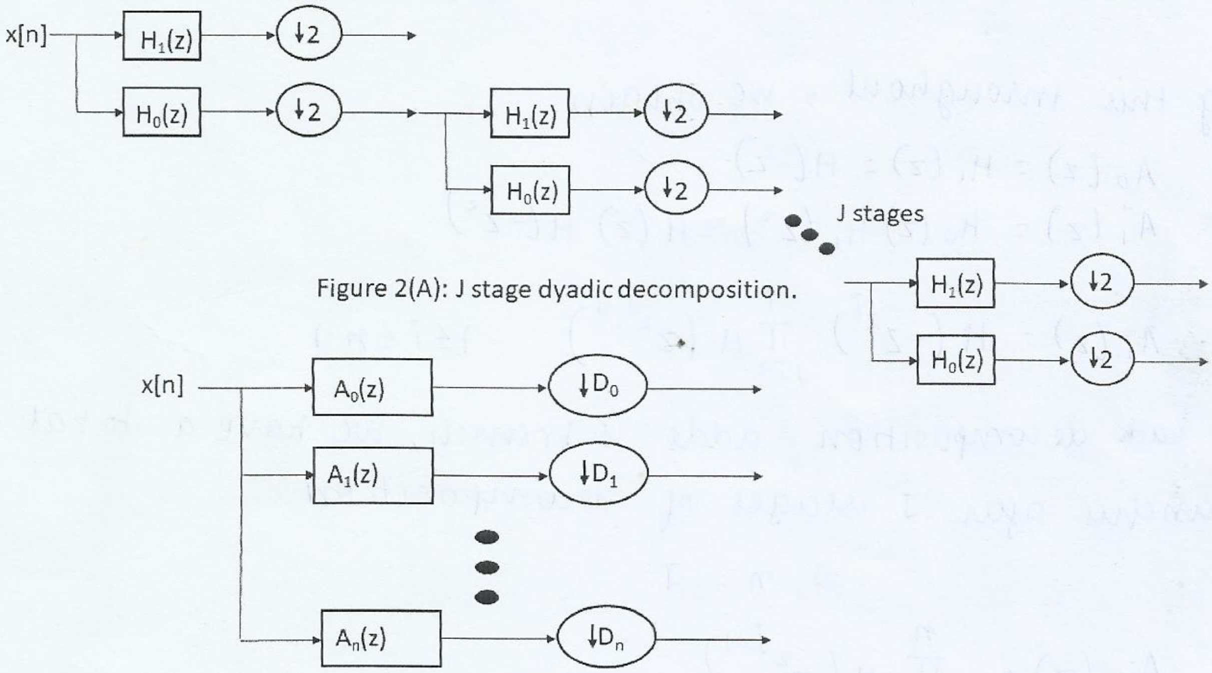


Figure 2(A): J stage dyadic decomposition.

Figure 2(B): Equivalent representation of Figure 2(A).

FIGURE 2. Subband coding of speech.

SOLUTION:

1) the original bit rate is given by:

R_b = sampling rate × ADC resolution

$= 16 \times 10^3$ samples per s × 12 bits per sample.

= 192 Kbps

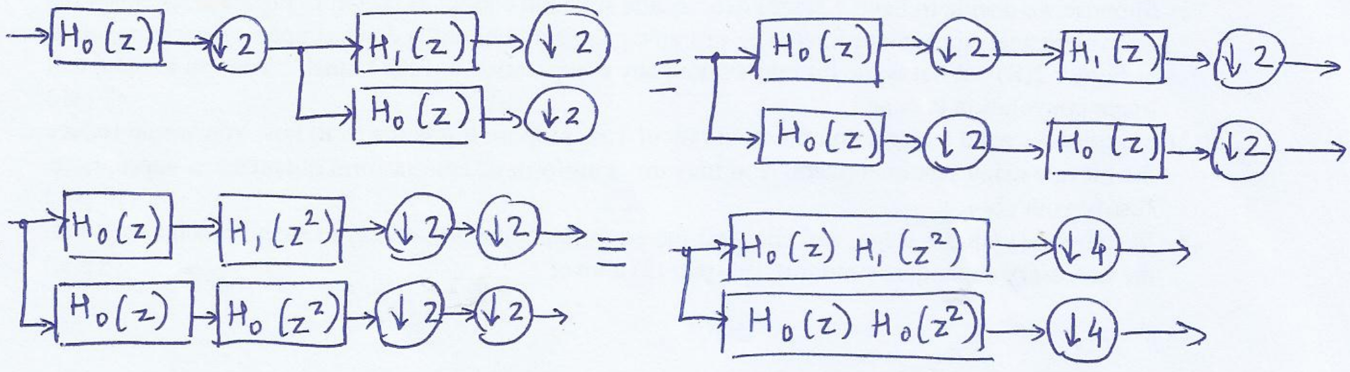
signal length = R_b × signal duration

= 120 s × 192 Kbps = 23.04×10^6 bits

(2) We consider $H_0(z) = H(z)$ and $H_1(z) = H(-z)$ (QMF)

We will use the following noble identity throughout:

consider the following schematic



using this throughout, we obtain.

$$A_0(z) = H_1(z) = H(-z)$$

$$A_1(z) = H_0(z)\, H_1(z^2) = H(z)\, H(-z^2)$$

$$\vdots$$

$$A_i(z) = H(-z^{2^i}) \prod_{j=1}^{i} H(z^{2^{j-1}}) \qquad 1 \le i \le n-1$$

As each decomposition adds 1 branch, we have a total of $J+1$ branches after J stages of decomposition

$$\Rightarrow n = J.$$

$$A_n(z) = \prod_{j=1}^{n} H(z^{2^{j-1}})$$

The decimator in each branch, by the above schematic, is obtained as:

$$D_i = 2^{i+1} \qquad 0 \le i \le n-1$$

$$D_n = 2^n.$$

Let us consider the orders of the filters. $H(z)$ has order N. $H(z^2)$ will have order $2N$. When we multiply (cascade) 2 filters their orders add up. Thus, $A_i = H(-z^{2^i}) \prod_{j=1}^{i} H(z^{2^{j-1}})$ will be of order $(2^i + \sum_{j=1}^{i} 2^{j-1})N$ $= (2^{i+1}-1)N$.

$\therefore$ For $i = 0$, order of $A_i = N$

$1 \le i \le n-1$, order of $A_i = (2^{i+1}-1)N$

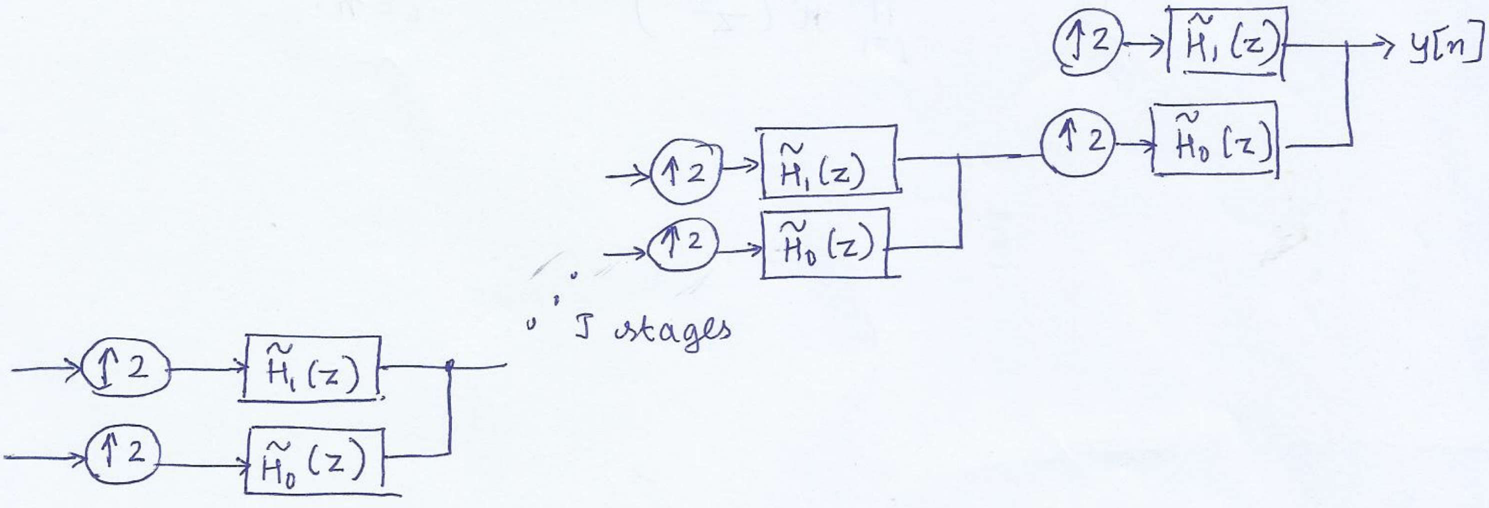
$i = n$, order of $A_i = (2^n - 1)N$.

the bit rate in each stage depends on the input bit rate and the decimator. If R is the input bit rate, then the bit rate in the branch corresponding to decimator D_i is $\frac{R}{D_i}$

$\Rightarrow$ branch bit rate: $R_i = \frac{R}{D_i} = \begin{cases} \frac{R}{2^{i+1}} & 0 \le i \le n-1 \\ \frac{R}{2^n} & i = n. \end{cases}$

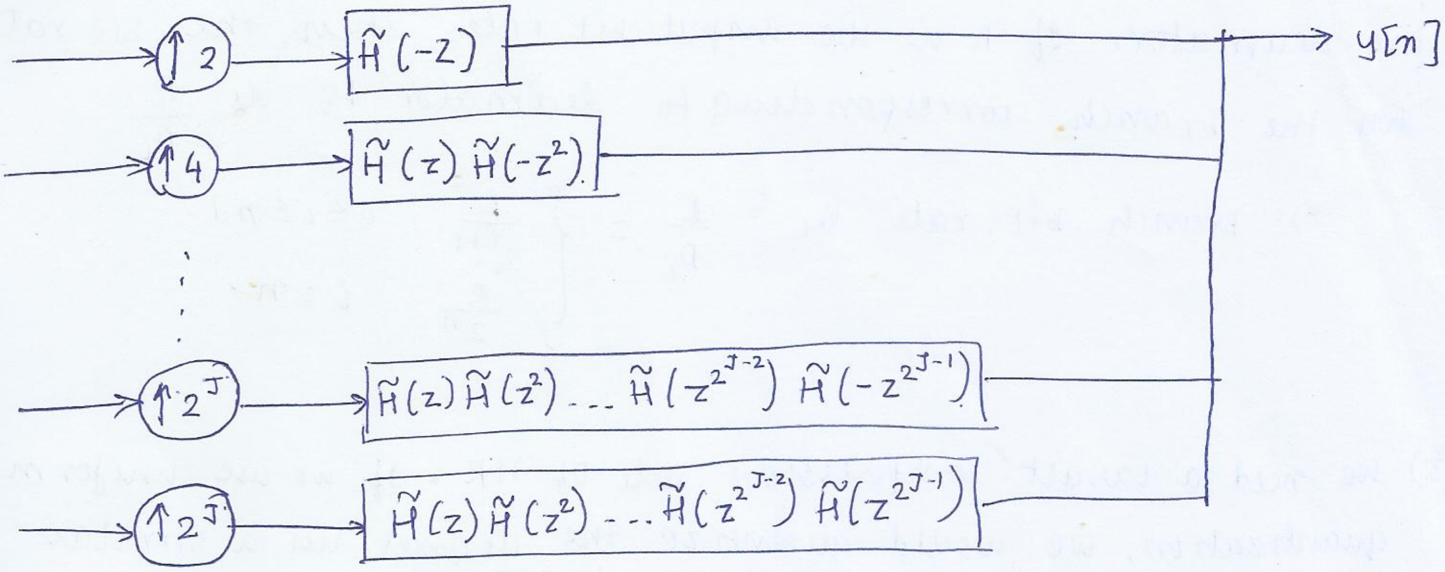
(3) we need a target compression rate of $1:R$. If we use uniform quantization, we would quantize the signal in a similar fashion in the low and high frequency bands. But it is known that in speech signals, the energy is more concentrated in the low frequency bands rather than the high frequency bands. Thus, using non uniform quantization we can quantize the high frequency bands at a higher rate assigning it a rate R_0 and proceed so on to assign rate $R_1, R_2, \ldots R_n$ such that $R_0 + R_1 + \ldots + R_n = R$.

(4) We can obtain the synthesis stage similar to the analysis stage. Let $\tilde{H}(z)$ be the FIR filter used in the synthesis stage. Let $\tilde{H}_1(z) = \tilde{H}(-z)$ and $\tilde{H}_0(z) = \tilde{H}(z)$

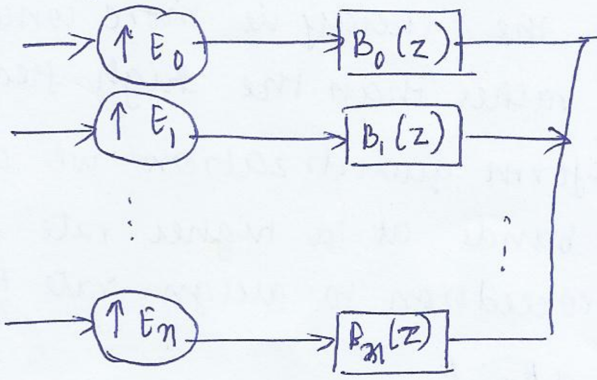


using the noble identity below, we can simply the above schematic

On simplification, we obtain,



comparing with.



we obtain
$$E_i = \begin{cases} 2^{i+1} & 0 \le i \le n-1 \\ 2^i & i = n \end{cases}$$

$$B_i(z) = \begin{cases} \tilde{H}(-z^{2^i}) \prod_{j=1}^{i} \tilde{H}(z^{2^{j-1}}) & 0 \le i \le n-1 \\ \prod_{j=1}^{n} \tilde{H}(-z^{2^{j-1}}) & i = n. \end{cases}$$