Generalized state space model
Linear discrete time models

Consider the linear discrete time model with transfer function for (p=q) case.

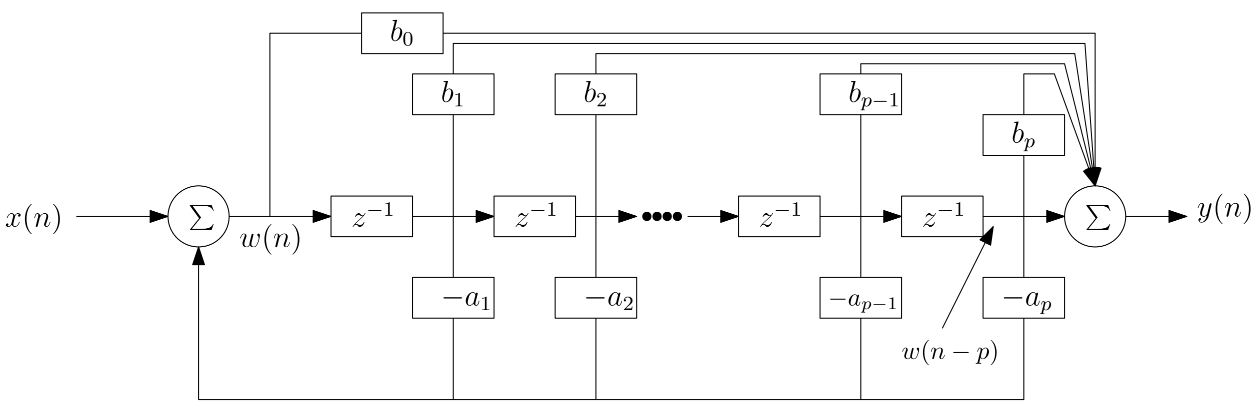
$$H(z) = \frac{\sum_{k=0}^{p} b_k z^{-k}}{1 + \sum_{k=1}^{p} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

Let us define two related transfer functions as follows

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{p} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

Let us form the signal flow graph for representing transfer functions above.

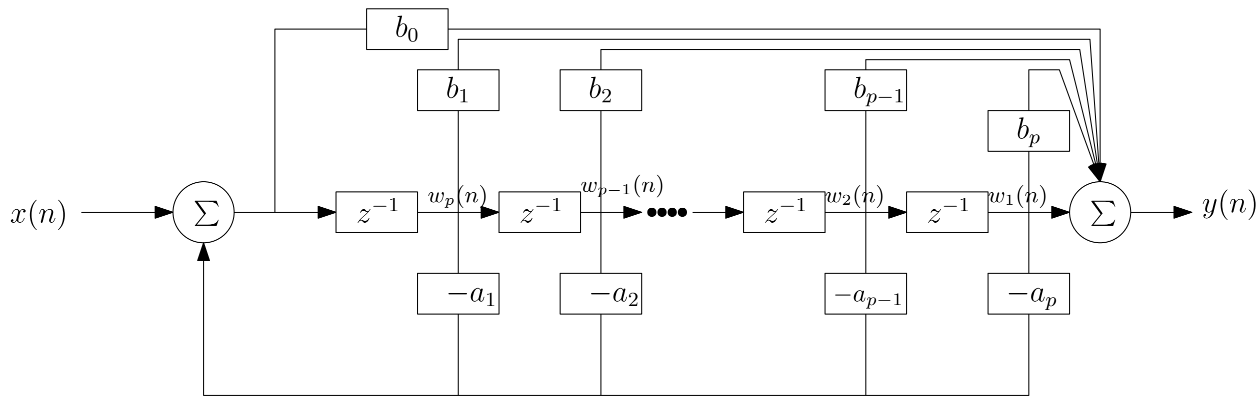


Define the state variables as follows:

$$\begin{aligned} w_p(n) &= w(n-1) \\ w_{p-1}(n) &= w(n-2) \\ &\vdots \\ w_1(n) &= w(n-p) \end{aligned}$$

As the signal $w(n)$ passes through the delay line, the state variables $[w_1(n), \ldots, w_p(n)]$ form a vector. The time to space mapping dictates that the signal in time can be transformed to a vector in space. The signal dynamics can be visualized as a **trajectory** as below.

$$\begin{aligned} w_1(n+1) &= w_2(n) \\ &\vdots \\ w_{p-1}(n+1) &= w_p(n) \\ w_p(n+1) &= x(n) - a_1 w_p(n) - a_2 w_{p-1}(n) - \cdots - a_p w_1(n) \end{aligned}$$



Let us form a state vector $\underline{W}(n) = [w_1(n), \ldots, w_p(n)]^{\mathrm{T}}$. Using this and the above expressions, we have

$$\underline{W}(n+1) = \mathbf{A}\underline{W}(n) + \mathbf{b}x(n)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & & & & \\ -a_p & -a_{p-1} & -a_{p-2} & -a_{p-3} & \ldots & -a_2 & -a_1 \end{bmatrix},$$

$$\mathbf{b} = [\underbrace{0, 0, \ldots, 0, 1}_{p \text{ elements}}]^{\mathrm{T}}$$

Similarly, one can do the math for expressing the output $y(n)$ through a sequence of equations below:

$$y(n) = b_0 w(n) + \sum_{k=1}^{p} b_k w_{p+1-k}(n)$$

$$y(n) = b_0 w_p(n+1) + \sum_{k=1}^{p} b_k w_{p+1-k}(n)$$

$$y(n) = b_0[x(n) - a_1 w_p(n) - a_2 w_{p-1}(n) - \cdots - a_p w_1(n)] + b_1 w_p(n) + b_2 w_{p-1}(n) + \cdots + b_p w_1(n)$$

$$y(n) = \sum_{k=1}^{p} [b_k - b_0 a_k] w_{p+1-k}(n) + b_0 x(n)$$

$$y(n) = \mathbf{c}^{\mathrm{T}}\underline{W}(n) + \mathbf{d}x(n)$$

where

$$\mathbf{c} = \begin{bmatrix} b_p - b_0 a_p \\ \vdots \\ b_1 - b_0 a_1 \end{bmatrix},$$

$$\mathbf{d} = b_0.$$