Indian Institute of Science

E9−252: Mathematical Methods and Techniques in Signal Processing
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Homework #1 Solutions, Fall 2017
Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late
Assigned date: Aug. 28th 2017 **Due date**: Sept. 4th 2017 by end of the day

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PROBLEM 1:
Can convolution operator be expressed as an inner product? Justify. (3 points)

Solution: Let f, g be two functions, then the convolution operation is defined as:

$$m(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

The inner product of two functions is defined as $\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)g(t)dt$. The convolution operation $m(k)$ can be written as the inner product of the functions $f(t)$ and $g(k-t)$.

$$m(k) = f(k) * g(k) = \langle f(t), g(k-t)\rangle$$

PROBLEM 2:
Define inner products of vectors defined over a complex field $\mathbb{C}$. For complex vectors x and y, compute $\langle x-y, x-y\rangle$ using the inner product defined. Derive the Cauchy-Schwarz inequality for complex vectors. (10 = 3+2+5 points)

Solution: A map $\langle\cdot,\cdot\rangle\, \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is an inner product of vectors defined over a complex vector space $\mathbb{C}^n$ if it satisfies the following properties:

a) $\langle \underline{x}, \underline{x}\rangle \geq 0$. $\langle \underline{x}, \underline{x}\rangle = 0$ iff $\underline{x} = 0$.

b) $\langle \underline{x}, \underline{y}\rangle = \overline{\langle \underline{x}, \underline{y}\rangle}$.

c) $\langle \underline{x}+\underline{y}, \underline{z}\rangle = \langle \underline{x}, \underline{z}\rangle + \langle \underline{y}, \underline{z}\rangle$ and $\langle \underline{x}, \underline{y}+\underline{z}\rangle = \langle \underline{x}, \underline{y}\rangle + \langle \underline{x}, \underline{z}\rangle$.

d) $\langle \alpha\underline{x}, \underline{y}\rangle = \alpha\langle \underline{x}, \underline{y}\rangle$ and $\langle \underline{x}, \alpha\underline{y}\rangle = \overline{\alpha}\langle \underline{x}, \underline{y}\rangle$.

For complex vectors x and y,

$$\begin{aligned}\langle \underline{x}-\underline{y}, \underline{x}-\underline{y}\rangle &= \langle \underline{x}, \underline{x}\rangle - \langle \underline{x}, \underline{y}\rangle - \langle \underline{y}, \underline{x}\rangle + \langle \underline{y}, \underline{y}\rangle \\ &= \langle \underline{x}, \underline{x}\rangle - \langle \underline{x}, \underline{y}\rangle - \overline{\langle \underline{x}, \underline{y}\rangle} + \langle \underline{y}, \underline{y}\rangle \\ &= \langle \underline{x}, \underline{x}\rangle - 2\,\mathrm{Re}(\langle \underline{x}, \underline{y}\rangle) + \langle \underline{y}, \underline{y}\rangle\end{aligned}$$

- Let us consider the following induced norm,

$$\begin{aligned}||x-\alpha y||^2 &= \langle \underline{x}-\alpha\underline{y}, \underline{x}-\alpha\underline{y}\rangle = \langle \underline{x}, \underline{x}\rangle - \langle \underline{x}, \alpha\underline{y}\rangle - \langle \alpha\underline{y}, \underline{x}\rangle + \langle \alpha\underline{y}, \alpha\underline{y}\rangle \\ &= \langle \underline{x}, \underline{x}\rangle - \overline{\alpha}\langle \underline{x}, \underline{y}\rangle - \alpha\langle \underline{y}, \underline{x}\rangle + |\alpha|^2\langle \underline{y}, \underline{y}\rangle\end{aligned}$$

As we need to find a bound, lets try to find the point where the minimum occurs. Differentiating with respect to $\overline{\alpha}$, we obtain,

$$\frac{\partial}{\partial\overline{\alpha}}(||x-\alpha y||^2) = 0 \Rightarrow -\langle \underline{x}, \underline{y}\rangle + \alpha\langle \underline{y}, \underline{y}\rangle = 0$$
$$\Rightarrow \alpha = \frac{\langle \underline{x}, \underline{y}\rangle}{\langle \underline{y}, \underline{y}\rangle}$$

Substituting the computed α in the induced inner product,

$$\begin{aligned}||x-\alpha y||^2 &= \langle \underline{x}, \underline{x}\rangle - \frac{\overline{\langle \underline{x}, \underline{y}\rangle}}{\langle \underline{y}, \underline{y}\rangle}\langle \underline{x}, \underline{y}\rangle - \frac{\langle \underline{x}, \underline{y}\rangle}{\langle \underline{y}, \underline{y}\rangle}\langle \underline{y}, \underline{x}\rangle + \frac{|\langle \underline{x}, \underline{y}\rangle|^2}{|\langle \underline{y}, \underline{y}\rangle|^2}\langle \underline{y}, \underline{y}\rangle \\ &= \langle \underline{x}, \underline{x}\rangle - \frac{\overline{\langle \underline{x}, \underline{y}\rangle}}{\langle \underline{y}, \underline{y}\rangle}\langle \underline{x}, \underline{y}\rangle\end{aligned}$$

As norm is non negative,

$$\begin{aligned} ||x-\alpha y||^2 &\geq 0 \\ \langle \underline{x}, \underline{x} \rangle - \frac{\overline{\langle \underline{x}, \underline{y} \rangle}}{\langle \underline{y}, \underline{y} \rangle} \langle \underline{x}, \underline{y} \rangle &\geq 0 \\ \langle \underline{x}, \underline{x} \rangle \langle \underline{y}, \underline{y} \rangle &\geq |\langle \underline{x}, \underline{y} \rangle|^2 \end{aligned}$$

PROBLEM 3:

a) Let $S_p = \{x : ||x||_p \leq 1\}$. Prove that $S_p \subset S_{p+1}$.

b) Prove that $\lim_{p\to\infty} \mathcal{L}_p = \mathcal{L}_\infty$.

(9 = 4+5 points)

Solution:

a) Let $x \in S_p$ and $x \in \mathbb{R}$, then,

$$\begin{aligned} & ||x||_p \leq 1 \quad \Rightarrow (\textstyle\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1 \\ \Rightarrow \quad & (\textstyle\sum_{i=1}^n |x_i|^p) \leq 1 \quad \Rightarrow |x_i|^p \leq 1 \\ \Rightarrow \quad & |x_i| \leq 1 \quad \Rightarrow |x_i|^{p+1} \leq |x_i|^p \\ \Rightarrow \quad & \left(\textstyle\sum_{i=1}^n |x_i|^{p+1}\right) \leq (\textstyle\sum_{i=1}^n |x_i|^p) \leq 1 \\ \Rightarrow \quad & \left(\textstyle\sum_{i=1}^n |x_i|^{p+1}\right)^{\frac{1}{p+1}} \leq 1 \Rightarrow \qquad x \in S_{p+1} \end{aligned}$$

Thus, $S_p \subset S_{p+1}$.

b) The $\mathcal{L}_p$ norm is given by,

$$\mathcal{L}_p(x) = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

Considering the limit as $p \to \infty$,

$$\begin{aligned} \lim_{p\to\infty} \mathcal{L}_p(x) &= \lim_{p\to\infty} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} = \lim_{p\to\infty} \left(\sum_{i=1}^n \left((\max_q |x_q|) \frac{|x_i|}{(\max_q |x_q|)}\right)^p\right)^{\frac{1}{p}} \\ &= \max_q |x_q| \lim_{p\to\infty} \left(\sum_{i=1}^n \left(\frac{|x_i|}{(\max_q |x_q|)}\right)^p\right)^{\frac{1}{p}} \end{aligned}$$

When $|x_i| \neq \max_q |x_q|$, $\frac{|x_i|}{(\max_q |x_q|)} < 1 \Rightarrow \lim_{p\to\infty} \left(\frac{|x_i|}{(\max_q |x_q|)}\right)^p = 0$. Thus,

$$\lim_{p\to\infty} \mathcal{L}_p(x) = \max_q |x_q| = \mathcal{L}_\infty(x)$$

PROBLEM 4: A function $f : X \to \mathbb{R}$ is called convex if

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \quad \forall x_1, x_2 \in X \text{ and } \alpha \in [0,1].$$

Examine if norm$(\cdot)$ is a convex function. (3 points)

Solution:

Let $x, y \in X$. Using triangle inequality property of the norm,

$$\begin{aligned} ||\alpha x + (1-\alpha)y|| &\leq ||\alpha x|| + ||(1-\alpha)y|| \quad \text{where } \alpha \in [0,1] \\ &= |\alpha| ||x|| + |(1-\alpha)| ||y|| \\ &= \alpha ||x|| + (1-\alpha) ||y|| \end{aligned}$$

Thus, norm is a convex function.