Indian Institute of Science

E9−252: Mathematical Methods and Techniques in Signal Processing

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Homework #4 Solutions, Fall 2017

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Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late

Assigned date: Sept. 18th 2017 **Due date**: Sept. 25th 2017 by end of the day

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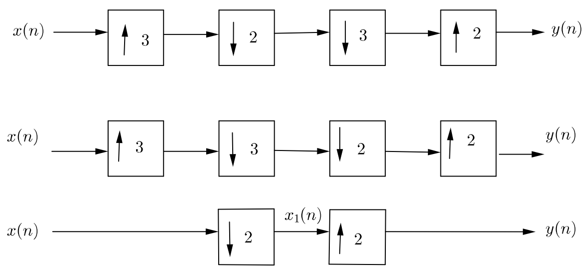
PROBLEM 1:
Problem 4.2

Solution:



$$X_1(z) = \frac{1}{2}\left(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right)$$
$$Y(z) = \frac{1}{2}\left(X(z) + X(-z)\right)$$
$$y(n) = \frac{1}{2}\left(x(n) + (-1)^n x(n)\right) = \frac{1}{2}\left(1 + (-1)^n\right) x(n) = \begin{cases} 0 & \text{n is odd} \\ x(n) & \text{n is even} \end{cases}$$

PROBLEM 2:
Problem 4.12

Solution:

a) $H(z) \rightarrow$ FIR filter of length 10

$$h(n) = \begin{cases} \left(\frac{1}{2}\right)^n & 0 \le n \le 9 \\ 0 & \text{else} \end{cases} = \left(\frac{1}{2}\right)^n (u(n) - u(n-10))$$

$$H(z) = Z\left(\left(\frac{1}{2}\right)^n u(n)\right) - Z\left(\left(\frac{1}{2}\right)^n u(n-10)\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n - \sum_{n=10}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n = \sum_{n=0}^{9}\left(\frac{1}{2}z^{-1}\right)^n$$
$$= 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \frac{1}{16}z^{-4} + \frac{1}{32}z^{-5} + \frac{1}{64}z^{-6} + \frac{1}{128}z^{-7} + \frac{1}{256}z^{-8} + \frac{1}{512}z^{-9}$$

$$E_0(z) = 1 + \frac{1}{4}z^{-1} + \frac{1}{16}z^{-2} + \frac{1}{64}z^{-3} + \frac{1}{256}z^{-4} = \sum_{n=0}^{4}\frac{1}{2^{2n}}z^{-n}$$

$$E_1(z) = \frac{1}{2} + \frac{1}{8}z^{-1} + \frac{1}{32}z^{-2} + \frac{1}{128}z^{-3} + \frac{1}{512}z^{-4} = \sum_{n=0}^{4}\frac{1}{2^{2n+1}}z^{-n}$$
$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

b) Let $H(z)$ be an IIR filter.

$$h(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(n-3)$$

$$H(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+3} z^{-n-3}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{2n+4} z^{-2n-4} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{2n+3} z^{-2n-3}$$

$$E_0(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n} z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{2n+4} z^{-n-2} = \sum_{n=0}^{\infty}\left(\left(\frac{1}{2}\right)^{2n} + z^{-2}\left(\frac{1}{3}\right)^{2n+4}\right) z^{-n}$$

$$= \mathcal{Z}\left(\left(\frac{1}{2}\right)^{2n} u(n)\right) + \frac{z^{-2}}{81}\mathcal{Z}\left(\left(\frac{1}{3}\right)^{2n} u(n)\right) = \mathcal{Z}\left(\left(\frac{1}{4}\right)^{n} u(n)\right) + \frac{z^{-2}}{81}\mathcal{Z}\left(\left(\frac{1}{9}\right)^{n} u(n)\right)$$

$$= \frac{4z}{4z-1} + \frac{z^{-2}}{81}\frac{9z}{9z-1} = \frac{4z}{4z-1} + \frac{z^{-1}}{9(9z-1)}$$

$$= \frac{4z(81z-9)+4-z^{-1}}{(4z-1)(81z-9)} = \frac{324z^2-36z+4-z^{-1}}{(4z-1)(81z-9)}$$

$$E_1(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n+1} z^{-n} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{2n+3} z^{-n-1} = \frac{1}{2}\mathcal{Z}\left(\left(\frac{1}{4}\right)^{n} u(n)\right) + \frac{z^{-1}}{27}\mathcal{Z}\left(\left(\frac{1}{9}\right)^{n} u(n)\right)$$

$$= \frac{1}{2}\frac{4z}{4z-1} + \frac{z^{-1}}{27}\frac{9z}{9z-1} = \frac{12z(9z-1)+2(4z-1)}{6(4z-1)(9z-1)}$$

$$= \frac{108z^2-12z+8z-2}{6(4z-1)(9z-1)} = \frac{108z^2-4z-2}{6(4z-1)(9z-1)}$$

PROBLEM 3:
Problem 4.14

Solution:

$$H(z) = \frac{1}{(1-2R\cos\theta z^{-1} + R^2 z^{-2})},\ R>0 \text{ and } \theta \text{ real}$$

$$z^{-1} = \frac{+2R\cos\theta \pm \sqrt{4R^2\cos^2\theta - 4R^2}}{2R^2} = \frac{1}{R}(\cos\theta \pm \sqrt{\cos^2\theta - 1}) = \frac{1}{R}(\cos\theta \pm i\sin\theta) = \frac{1}{R}e^{\pm j\theta} \Rightarrow z = Re^{\pm j\theta}$$

$$H(z) = \frac{1}{R^2\left(z^{-1} - \frac{e^{j\theta}}{R}\right)\left(z^{-1} - \frac{e^{-j\theta}}{R}\right)} = \frac{z^2}{(R-e^{j\theta}z)(R-e^{-j\theta}z)} = \frac{z^2}{(z-Re^{-j\theta})(z-Re^{j\theta})}$$

Let $H(Z) = E_0(z^2) + z^{-1}E_1(z^2)$, then,

$$H(-Z) = E_0(z^2) - z^{-1}E_1(z^2) \Rightarrow E_0(z^2) = \frac{H(z)+H(-z)}{2} \text{ and } E_1(z^2) = z\frac{H(z)-H(-z)}{2}$$

$$E_0(z^2) = \frac{z^2}{2}\left(\frac{1}{(z-Re^{-j\theta})(z-Re^{j\theta})} + \frac{1}{(z+Re^{-j\theta})(z+Re^{j\theta})}\right) = \frac{z^2(z^2+R^2)}{z^4 - 2R^2z^2\cos 2\theta + R^4}$$

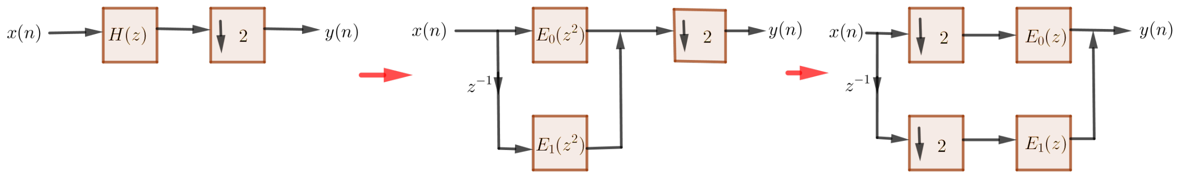
$$E_1(z^2) = \frac{z^3}{2}\left(\frac{1}{(z-Re^{-j\theta})(z-Re^{j\theta})} - \frac{1}{(z+Re^{-j\theta})(z+Re^{j\theta})}\right) = \frac{2Rz^4\cos 2\theta}{z^4 - 2R^2z^2\cos 2\theta + R^4}$$

$$E_0(z) = \frac{z(z+R^2)}{z^2 - 2R^2z\cos 2\theta + R^4}, \quad E_1(z) = \frac{2Rz^2\cos 2\theta}{z^2 - 2R^2z\cos 2\theta + R^4}.$$

PROBLEM 2:
Devise an efficient architecture to exploit the mirror/symmetric properties of polyphase components in decimation and interpolation filters.

Solution:

For decimation filter, we use Type 1 polyphase representation $H(z) = \sum_{i=0}^{M-1} z^{-i} E_i(z^M)$ to obtain the polyphase components $E_i(z)$'s. For $M = 2$, we obtain the following:



Now, let $H(z)$ be a linear phase filter of length N. Let $H(z) = \sum_{i=0}^{N-1} h_i z^{-i}$ with $h_i = h_{N-1-i}$.

If N is odd,

$$E_0(z^2) = \sum_{i=0}^{\frac{N-1}{2}} h_{2i} z^{-2i}$$

$$E_1(z^2) = \sum_{i=0}^{\frac{N-1}{2}-1} h_{2i+1} z^{-2i}$$

Thus, $h_{2i} = h_{N-1-2i} = h_{2(\frac{N-1}{2}-i)}$ and $h_{2i+1} = h_{N-1-2i-1} = h_{2(\frac{N-1}{2}-1-i)+1}$.

Thus, the filters are again symmetric and we decompose it further.

If N is even,

$$E_0(z^2) = \sum_{i=0}^{\frac{N}{2}-1} h_{2i} z^{-2i}$$

$$E_1(z^2) = \sum_{i=0}^{\frac{N}{2}-1} h_{2i+1} z^{-2i}$$

Thus, $h_{2i} = h_{N-1-2i} = h_{2(\frac{N}{2}-1-i)+1}$ and $h_{2i+1} = h_{N-1-2i-1} = h_{2(\frac{N}{2}-1-i)}$.

Thus, the filters are mirrors of each other.

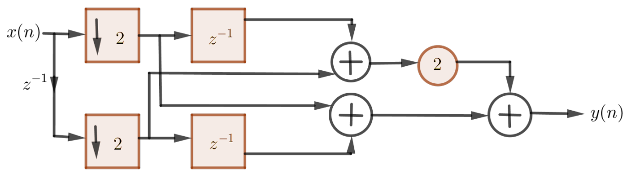
$$E_1(z^2) = \sum_{i=0}^{\frac{N}{2}-1} h_{2i+1} z^{-2i} = \sum_{i=0}^{\frac{N}{2}-1} h_{2(\frac{N}{2}-1-i)} z^{-2i}$$

$$= z^{(-N+2)} \sum_{i=0}^{\frac{N}{2}-1} h_{2(\frac{N}{2}-1-i)} z^{2(\frac{N}{2}-1-i)} = z^{(-N+2)} E_0(z^{-2})$$

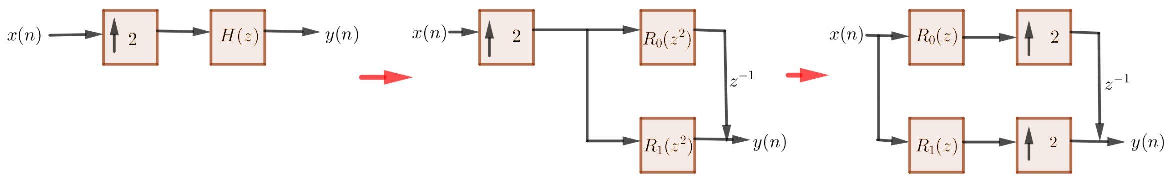
$$E_1(z) = z^{-(\frac{N}{2}-1)} E_0(z^{-1})$$

We exploit the mirror property by designing only one filter and at the input of the filter adding the respective inputs where the input from upper branch is in the reverse order when compared

to that of lower branch.
Let us consider an example. Let $N = 4$ and $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$. By Type 1 decomposition, $E_0(z) = 1 + 2z^{-1}$ and $E_1(z) = 2 + z^{-1}$. As they are mirrors of each other, we can use one filter where the inputs from the two branches are added accordingly and given to a linear system as follows: Similarly for interpolation, we use Type 2 polyphase representation



$H(z) = \sum_{i=0}^{M-1} z^{-(M-1-i)} R_i(z^M)$ to obtain the polyphase components $R_i(z^M)$'s. For $M = 2$, we obtain the following:



When N is odd, we obtain symmetric filters which are further deomposed. When N is even, the relation between $R_0(z)$ and $R_1(z)$ is $R_0(z) = z^{-(\frac{N}{2}-1)} R_1(z^{-1})$. As the interpolation filter in each branch is followed by expander and there is a delay in the first branch, we obtain the output of the system as,

$$\begin{aligned} Y(z) &= (z^{-1}R_0(z^2) + R_1(z^2))X(z^2) \\ &= z^{-1}R_0(z^2)X(z^2) + R_1(z^2)X(z^2) \end{aligned}$$

Using the filter $R_1(z)$ in the lower branch in one clock cycle and the same filter with the input order reversed in the next clock cycle to obtain the output in the upper branch, we obtain the output $y(n)$. This is equivalent to the final decomposition in the figure as the expander would introduce zeros while expanding and it together with the delay would be equivalent to interleaving the outputs from the component filters. If we consider the same filter $H(z) = 1+2z^{-1}+2z^{-2}+z^{-3}$, by Type 2 decomposition, we obtain $R_0(z) = 2+z^{-1}$ and $R_1(z) = 1+2z^{-1}$. Thus, using the filter for the upper branch and lower branch in alternate cycles, we obtain: