Indian Institute of Science

E9−252: Mathematical Methods and Techniques in Signal Processing

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Homework #7 Solutions, Fall 2017

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Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late

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PROBLEM 1:

Derive wavelet decomposition of a signal using m-adic Haar wavelets. (10 points)

Solution:

First let us obtain the m-adic Haar wavelets. Let the basis of V_j be $\{\phi(m^j t - k)\}_{k=-\infty}^{\infty}$, where,

$$\phi(m^j t - k) = \begin{cases} 1 & \dfrac{k}{m^j} \leq t < \dfrac{k+1}{m^j} \\ 0 & \text{else} \end{cases}$$

Note that $\phi(m^j t - k) = \sum_{a=0}^{m-1} \phi(m^{j+1}t - mk + a)$. As it needs m signals in V_{j+1} to obtain V_j, thus the dimensionality of V_{j+1} with respect to V_j is m. Thus, we need $m-1$ wavelets to obtain V_{j+1} using V_j. Let us define the wavelets in W_0 as follows:

$$\psi_i(t) = \begin{cases} x & 0 \leq t < \dfrac{i}{m} \\ y_i & \dfrac{i}{m} \leq t < \dfrac{i+1}{m} \\ 0 & \text{else} \end{cases} \tag{1}$$

As we need the wavelets to be orthogonal to each other, consider $\psi_i(t)$ and $\psi_j(t)$, where $i < j$, then,

$$\begin{aligned} \int_{-\infty}^{\infty} \psi_i(t)\psi_j(t)dt &= 0 \\ \implies \frac{x^2 i}{m} + \frac{x y_i}{m} &= 0 \\ \implies \frac{xi + y_i}{m} &= 0 \end{aligned}$$

As we need the wavelets to be orthogonal to $\phi(t)$,

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t)\psi_i(t)dt &= 0 \\ \implies \frac{xi}{m} + \frac{y_i}{m} &= 0 \\ \implies \frac{xi + y_i}{m} &= 0 \end{aligned}$$

Thus, considering $x = 1 \implies y_i = -i$. Thus,

$$\psi_i(t) = \begin{cases} 1 & 0 \leq t < \dfrac{i}{m} \\ -i & \dfrac{i}{m} \leq t < \dfrac{i+1}{m} \\ 0 & \text{else} \end{cases}$$

These form the basis of the wavelet space W_0. Note that $|\psi_j(t)|^2 = \int_{-\infty}^{\infty} \psi_j^2(t)dt = \frac{j(j+1)}{m}$. Let us denote this by α_j^2. As $\phi(t)$ and $\{\psi_j(t)\}_{j=1}^{m-1}$ are orthogonal and lie in V_1, they form a basis for V_1 as it has dimension of m. Thus, we can represent $\phi(mt)$ as follows:

$$\phi(mt) = a_0\phi(t) + \sum_{i=1}^{m-1} a_i\psi_i(t).$$

where the coefficients are obtained as follows:

$$\begin{aligned} a_0 &= \langle \phi(mt), \phi(t) \rangle \\ a_i &= \frac{1}{\alpha_i} \langle \phi(mt), \psi_i(t) \rangle \end{aligned}$$

In general,

$$\langle \phi(mt-j), \psi_i(t) \rangle = \begin{cases} \frac{1}{m} & j < i \\ \frac{-i}{m} & j = i \\ 0 & j > i \end{cases}$$

Thus, $a_i = \frac{1}{m\alpha_i}$ As we have obtained V_1 from V_0 and W_0, we can similarly obtain the higher resolution subspaces and obtain a wavelet decomposition.

<u>Generalization:</u> The wavelets in W_j are given by,

$$\psi_i(m^j t) = \begin{cases} 1 & 0 \le t < \frac{i}{m^{j+1}} \\ -i & \frac{i}{m^{j+1}} \le t < \frac{i+1}{m^{j+1}} \\ 0 & \text{else} \end{cases}$$

Note that $|\psi_j(t)|^2 = \int_{-\infty}^{\infty} \psi_j^2(t)dt = \frac{j(j+1)}{m^j}$. Let us denote this by α_j^2. As $\phi(m^j t)$ and $\{\psi_k(m^j t)\}_{k=1}^{m-1}$ are orthogonal and lie in V_{j+1}, they form a basis for it as it has dimension of m. Thus, we can represent $\phi(m^{j+1}t)$ as follows:

$$\phi(m^{j+1}t) = a_0\phi(m^j t) + \sum_{i=1}^{m-1} a_i\psi_i(m^j t).$$

where the coefficients are obtained as follows:

$$\begin{aligned} a_0 &= \frac{1}{m^j} \\ a_i &= \frac{1}{m^j \alpha_i} \end{aligned}$$

PROBLEM 2:

Let W_j be the space of all functions with basis $\psi(2^j t - k)$ where $k \in \mathbb{Z}$. Prove $V_{j+1} = V_j \oplus W_j$. (5 points)

Solution:

Any element in V_{j+1} is given by:

$$f(t) = \sum_{l=-\infty}^{\infty} a_l \phi(2^{j+1}t - l)$$

The basis function is given by:

$$\phi(2^{j+1}t-k)=\begin{cases}1 & \frac{k}{2^{j+1}}\le t<\frac{k+1}{2^{j+1}}\\ 0 & \text{else}\end{cases}$$

Let us consider the basis functions $\phi(2^jt-k)$ and $\psi(2^jt-k)$ of V_j and W_j respectively, then,

$$\begin{aligned}\phi(2^jt-k) &= \begin{cases}1 & \frac{k}{2^j}\le t<\frac{k+1}{2^j}\\ 0 & \text{else}\end{cases}\\ \psi(2^jt-k) &= \begin{cases}1 & \frac{k}{2^j}\le t<\frac{k}{2^j}+\frac{1}{2^{j+1}}\\ -1 & \frac{k}{2^j}+\frac{1}{2^{j+1}}\le t<\frac{k+1}{2^{j+1}}\\ 0 & \text{else}\end{cases}\end{aligned}$$

Thus, if l is odd,

$$\phi(2^{j+1}-l)=\frac{\phi(2^jt-\frac{l-1}{2})-\psi(2^jt-\frac{l-1}{2})}{2}$$

Thus, if l is even,

$$\phi(2^{j+1}-l)=\frac{\phi(2^jt-\frac{l}{2})+\psi(2^jt-\frac{l}{2})}{2}$$

Thus,

$$\begin{aligned}f(t) &= \sum_{l=-\infty}^{\infty}a_l\phi(2^{j+1}t-l)=\sum_{l=-\infty}^{\infty}a_{2l}(\phi(2^jt-l)+(\psi(2^jt-l))+\sum_{l=-\infty}^{\infty}a_{2l+1}(\phi(2^jt-l)-(\psi(2^jt-l))\\ &= \sum_{l=-\infty}^{\infty}(a_{2l}+a_{2l+1})\phi(2^jt-l)+(a_{2l}-a_{2l+1}\psi(2^jt-l))\end{aligned}$$

Thus, every signal can be expressed in basis of V_j and ψ_j which are orthogonal to each other. Thus, the direct sum of the two spaces add upto V_{j+1}.

PROBLEM 3:
Obtain the Haar wavelet decomposition for the signal $f(t)$ using the Haar basis. Indicate the signal dimension at each subspace. Sketch the waveforms explicitly at each subspace. Obtain the reconstructed signal in functional form after nulling out any spike of $(1/8)^{\text{th}}$ unit of time. Analyze using Fourier Transform. How much of energy is lost in the recovered signal?(10 points)

$$f(t)=\begin{cases}3 & 0\le t<\frac{1}{4}\\ -1 & \frac{1}{4}\le t<\frac{3}{8}\\ 2 & \frac{3}{8}\le t<\frac{5}{8}\\ 0 & \frac{5}{8}\le t<1\end{cases}$$

Solution:
The maximum resolution of the function is $\left(\frac{1}{8}\right)^{\text{th}}$ unit of time. The function can be written

Level	Signal Dimension
1	2
2	4
3	5
4	5

Table 1: Signal Dimension

as:

$$
\begin{aligned}
f(t) &= 3\phi(8t) + 3\phi(8t-1) - \phi(8t-2) + 2\phi(8t-3) + 2\phi(8t-4) \\
&= \frac{3}{2}(\phi(4t)+\psi(4t)) + \frac{3}{2}(\phi(4t)-\psi(4t)) - \frac{1}{2}(\phi(4t-1)+\psi(4t-1)) + (\phi(4t-1)-\psi(4t-1)) + (\phi(4t- \\
&= 3\phi(4t) + \frac{1}{2}\phi(4t-1) - \frac{3}{2}\psi(4t-1) + \phi(4t-2) + \psi(4t-2) \\
&= \frac{3}{2}(\phi(2t)+\psi(2t)) + \frac{1}{4}(\phi(2t)-\psi(2t)) + \frac{1}{2}(\phi(2t-1)+\psi(2t-1)) - \frac{3}{2}\psi(4t-1) + \psi(4t-2) \\
&= \frac{7}{4}\phi(2t) + \frac{5}{4}\psi(2t) + \frac{1}{2}(\phi(2t-1)+\psi(2t-1)) - \frac{3}{2}\psi(4t-1) + \psi(4t-2) \\
&= \frac{7}{8}(\phi(t)+\psi(t)) + \frac{5}{4}\psi(2t) + \frac{1}{2}(\frac{1}{2}(\phi(t)-\psi(t)) + \psi(2t-1)) - \frac{3}{2}\psi(4t-1) + \psi(4t-2) \\
&= \frac{9}{8}\phi(t) + \frac{5}{8}\psi(t) + \frac{5}{4}\psi(2t) + \psi(2t-1)) - \frac{3}{2}\psi(4t-1) + \psi(4t-2)
\end{aligned}
$$

The signal dimension at each level is: After nulling out the $\left(\frac{1}{8}\right)^{\text{th}}$ spike but suppressing $\psi(4t-1)$ and $\psi(4t-2)$, we obtain,

$$
\begin{aligned}
g(t) &= \frac{9}{8}\phi(t) + \frac{5}{8}\psi(t) + \frac{5}{4}\psi(2t) + \frac{1}{2}\psi(2t-1)) \\
&= \frac{9}{8}\phi(4t) + \frac{9}{8}\phi(4t-1) + \frac{9}{8}\phi(4t-2) + \frac{9}{8}\phi(4t-3) + \frac{5}{8}\phi(4t) + \frac{5}{8}\phi(4t-1) - \frac{5}{8}\phi(4t-2) \\
&\quad -\frac{5}{8}\psi(4t-3) + \frac{5}{4}\phi(4t) - \frac{5}{4}\phi(4t-1) + \frac{1}{2}\phi(4t-2) - \frac{1}{2}\phi(4t-3) \\
&= 3\phi(4t) + \frac{1}{2}\phi(4t-1) + \phi(4t-2) \\
&= \begin{cases} 3 & 0 \le t < \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < \frac{3}{4} \\ 0 & \frac{3}{4} \le t < 1 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{Energy lost} &= \int_{-\infty}^{\infty} f^2(t) - g^2(t)dt \\
&= (\frac{1}{4}\times 3^2 + \frac{1}{8}\times(-1)^2 + \frac{1}{4}\times 2^2 - (\frac{1}{4}\times 3^2 + \frac{1}{4}\times\left(\frac{1}{2}\right)^2 + \frac{1}{4}\times 1^2) \\
&= \frac{27}{8} - \frac{41}{16} = \frac{13}{16}
\end{aligned}
$$

Fourier Analysis:

$$
G(f) = \frac{3}{4}\text{sinc}\left(\frac{f}{4}\right)\text{e}^{\frac{-j2\pi f}{8}} + \frac{1}{8}\text{sinc}\left(\frac{f}{4}\right)\text{e}^{\frac{-j6\pi f}{8}} + \frac{1}{4}\text{sinc}\left(\frac{f}{4}\right)\text{e}^{\frac{-j10\pi f}{8}}
$$