Indian Institute of Science
E9−207: Basics of Signal Processing
Instructor: Shayan G. Srinivasa
Homework #3, Spring 2018
Solutions prepared by Arijit, Chaitanya and Ankur

PROBLEM 1:
(a)Let $H(z) = \frac{1+az^{-1}}{a+z^{-1}}$, $a \in \mathbb{R}$. Write down the expressions for the Type 1 polyphase components (with $M = 2$). What can you say about $H(z)$ for various values of a? (4 points)
(b)Let $H(z) = \frac{1}{1-2R\sin\theta z^{-1}+R^2z^{-2}}$ with $R > 0$ and $\theta \in \mathbb{R}$. Find the Type 1 polyphase components for $M = 2$. (4 points)
Solution:
Writing the polyphase decomposition of $H(z)$:

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

(a)

$$\begin{aligned} H(z) &= \frac{1+az^{-1}}{a+z^{-1}}, \quad a \in \mathbb{R} \\ &= \frac{(1+az^{-1})(a-z^{-1})}{a^2-z^{-2}} = \frac{a(1-z^{-2})}{a^2-z^{-2}} + z^{-1}\frac{a^2-1}{a^2-z^{-2}}. \\ \therefore E_0(z) &= \frac{a(1-z^{-1})}{a^2-z^{-1}} \text{ and } E_1(z) = \frac{a^2-1}{a^2-z^{-1}}. \end{aligned}$$

This gives the type-I decomposition of $H(z)$.

$$|H(e^{i\omega})| = 1 \Rightarrow \text{ It is an all pass filter } \forall a.$$

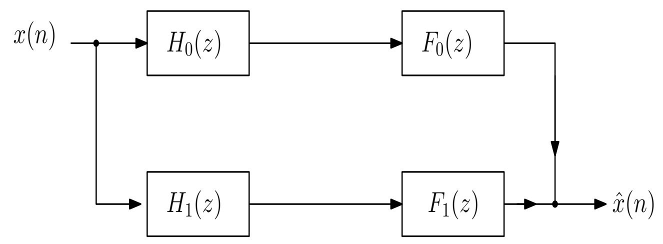
For $a = 0$, $H(z) = z \Rightarrow$ It is a non causal filter . For $a > 1$, $H(z)$ is stable and for $a < 1$, $H(z)$ is unstable.

(b)

$$\begin{aligned} H(z) &= \frac{1}{1-2R\sin\theta z^{-1}+R^2z^{-2}}, \quad R > 0, \theta \in \mathbb{R} \\ &= \frac{(1+R^2z^{-2})+2R\sin\theta z^{-1}}{(1+R^2z^{-2})^2-4R^2\sin^2\theta z^{-2}} = \frac{1+R^2z^{-2}}{1+2R^2\cos 2\theta z^{-2}+R^4z^{-4}} + z^{-1}\frac{2R\sin\theta}{1+2R^2\cos 2\theta z^{-2}+R^4z^{-4}}. \\ \therefore E_0(z) &= \frac{1+R^2z^{-1}}{1+2R^2\cos 2\theta z^{-1}+R^4z^{-2}} \text{ and } E_1(z) = \frac{2R\sin\theta}{1+2R^2\cos 2\theta z^{-1}+R^4z^{-2}}. \end{aligned}$$

This gives the type-I decomposition of $H(z)$.

PROBLEM 2:
Consider the analysis/synthesis system shown below:



(a) Let the analysis filters be $H_0(z) = 1 + 3z^{-1} + \frac{1}{3}z^{-2} + z^{-3}$ and $H_1(z) = H_0(-z)$. Find causal stable IIR filters $F_0(z)$ and $F_1(z)$ such that $\hat{x}(n)$ agrees with $x(n)$ except for a possible delay and (non zero) scale factor.(5 points)
(b) Let the analysis filters be $H_0(z) = 1 + z^{-1} + 3z^{-2} + z^{-3} + z^{-4}$ and $H_1(z) = H_0(-z)$. Find causal stable FIR filters $F_0(z)$ and $F_1(z)$ such that $\hat{x}(n)$ agrees with $x(n)$ except for a possible delay and (non zero) scale factor. (5 points)
Solution:
From the block diagram, we observe that

$$\hat{X}(z) = \big(H_0(z)F_0(z) + H_1(z)F_1(z)\big)X(z)$$

Writing the polyphase decomposition of $H_0(z)$:

$$\begin{aligned} H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2) \\ H_1(z) &= H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2) \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= E_0(z^2)\big(F_0(z) + F_1(z)\big) + z^{-1}E_1(z^2)\big(F_0(z) - F_1(z)\big) \end{aligned}$$

(a)

$$\begin{aligned} H_0(z) &= 1 + 3z^{-1} + \frac{1}{3}z^{-2} + z^{-3} \\ E_0(z^2) &= 1 + \frac{1}{3}z^{-2} \\ z^{-1}E_1(z^2) &= 3z^{-1} + z^{-3} \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= \big(1 + \frac{1}{3}z^{-2}\big)\big(F_0(z) + F_1(z)\big) + \big(3z^{-1} + z^{-3}\big)\big(F_0(z) - F_1(z)\big) \end{aligned}$$

We can choose $F_0(z) = F_1(z) = \frac{1}{2(1+\frac{1}{3}z^{-2})} \Rightarrow \frac{\hat{X}(z)}{X(z)} = 1$. Note that $F_0(z)$ and $F_1(z)$ are causal and stable because the poles are located at $\pm j\frac{1}{\sqrt{3}}$(inside the unit circle).

(b)

$$\begin{aligned} H_0(z) &= 1 + z^{-1} + 3z^{-2} + z^{-3} + z^{-4} \\ E_0(z^2) &= 1 + 3z^{-2} + z^{-4} \\ z^{-1}E_1(z^2) &= z^{-1} + z^{-3} = z^{-1}(1 + z^{-2}) \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= \big(1 + 3z^{-2} + z^{-4}\big)\big(F_0(z) + F_1(z)\big) + z^{-1}\big(1 + z^{-2}\big)\big(F_0(z) - F_1(z)\big) \\ &= \big((1 + z^{-2})^2 + z^{-2}\big)\big(F_0(z) + F_1(z)\big) + z^{-1}\big(1 + z^{-2}\big)\big(F_0(z) - F_1(z)\big) \\ &= (1 + z^{-2})^2(F_0(z) + F_1(z)) + z^{-2}(F_0(z) + F_1(z)) + z^{-1}\big(1 + z^{-2}\big)\big(F_0(z) - F_1(z)\big) \end{aligned}$$

We can choose $F_0(z) + F_1(z) = z^{-1}$ and $F_0(z) + F_1(z) = -(1 + z^{-2})$.

$$\Rightarrow \frac{\hat{X}(z)}{X(z)} = z^{-1}(1 + z^{-2})^2 + z^{-3} - z^{-1}(1 + z^{-2})^2 = z^{-3}.$$

Therefore, perfect reconstruction is possible with the following causal FIR filters:

$$F_0(z) = \frac{1}{2}(-1 + z^{-1} - z^{-2});$$
$$F_1(z) = \frac{1}{2}(1 + z^{-1} + z^{-2}).$$

PROBLEM 3:
Let $H_0(z) = \frac{1+2z^{-1}}{2}$. Find the real coefficient causal FIR filter $H_1(z)$ such that the pair $(H_0(z), H_1(z))$ is power complementary. Are these filters also all pass complementary? (6 points)
Solution:
Since $H_0(z)$ and $H_1(z)$ are power complementary, we have:

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = K \text{ (some constant)}$$
$$|1/2 + e^{-j\omega}|^2 + |H_1(e^{j\omega})|^2 = K$$
$$(1/2 + \cos\omega)^2 + \sin^2\omega + |H_1(e^{j\omega})|^2 = K$$
$$\frac{1}{4} + 1 + \cos\omega + |H_1(e^{j\omega})|^2 = K$$

By choosing $H_1(z) = \frac{1}{2} - z^{-1}$, we get

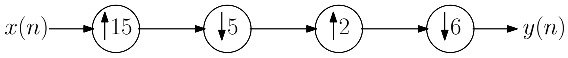
$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = \frac{5}{4} + \frac{5}{4} = \frac{5}{2}.$$

We check for all pass complementarity:

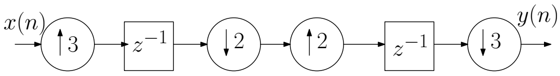
$$H_0(z) + H_1(z) = \frac{1}{2} + z^{-1} + \frac{1}{2} - z^{-1} = 1.$$

Therefore, $(H_0(z), H_1(z))$ are all pass complementary.

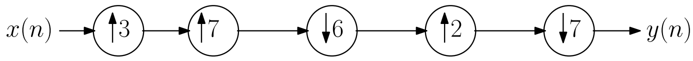
PROBLEM 4:
Simplify the following multirate systems shown below as best as you can. Obtain the z-transform of the output signal in terms of that of the input signal. ($3 \times 4 = 12$ points)



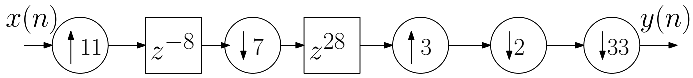
(a)



(b)

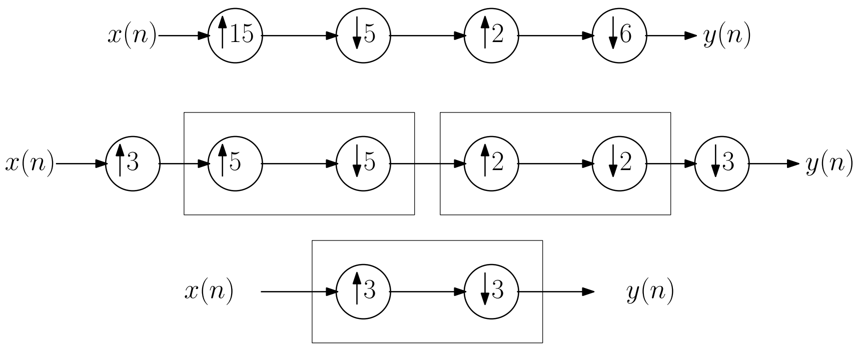


(c)

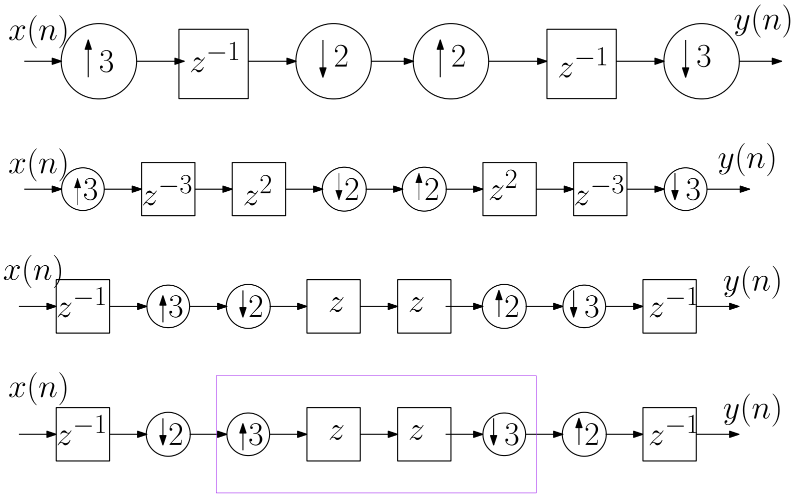


(d)

Solution:
(a) $y(n) = x(n)$.

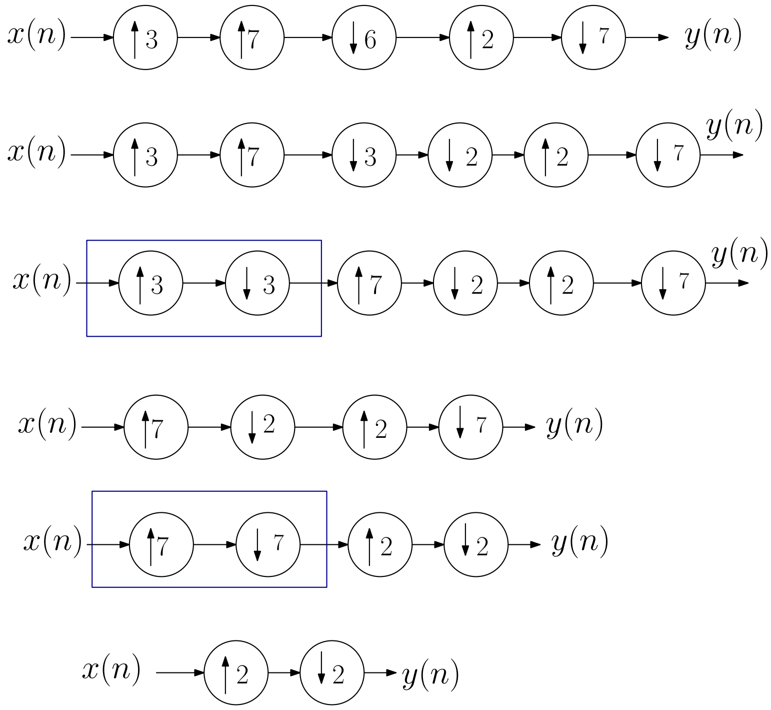


(b) $y(n) = 0$

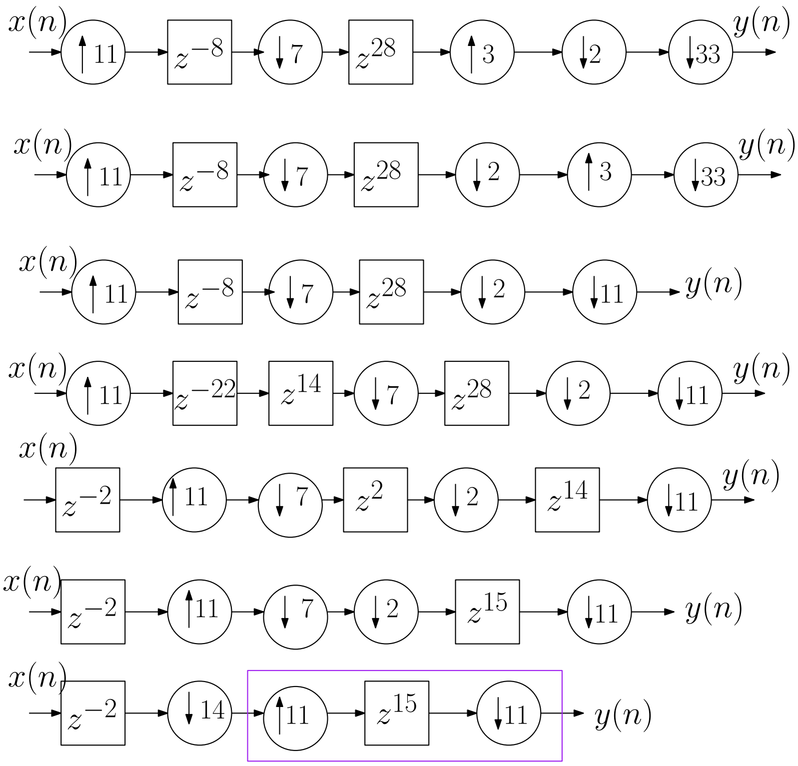


This block produces zeros for all input sequences

(c)$Y(z) = \frac{1}{2}(X(z) + X(-z))$.



(d) $y(n) = 0$



This block produces all zeros for all inputs

PROBLEM 5:
Prove that decimation by M followed by expansion by L can be interchanged if L and M are relatively prime. You must prove this result in the time and frequency domain representations. (10 points)
Solution:
(Frequency domain analysis) From Figure 2,

$$\begin{aligned} X_1(z) &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ \Longrightarrow Y_1(z) &= X_1\left(z^L\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i}{M}}\right). \qquad (1) \end{aligned}$$

Similarly,

$$\begin{aligned} X_2(z) &= X\left(z^L\right) \\ Y_2(z) &= \frac{1}{M}\sum_{i=0}^{M-1} X_2\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right)^L\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i L}{M}}\right). \qquad (2) \end{aligned}$$

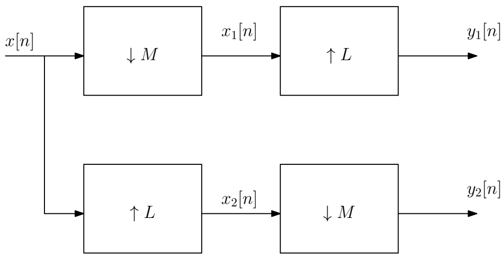


Figure 2: Comparing the outputs by changing the order of decimator and upsampler.

To prove that $Y_1(z) = Y_2(z)\,\forall X(z)$, it is necessary and sufficient to satisfy the following condition:

$$\begin{aligned} \left\{X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i L}{M}}\right) \mid i = 0,1,\cdots,M-1\right\} &= \left\{X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i}{M}}\right) \mid i = 0,1,\cdots,M-1\right\} \forall X(z) \\ i.e., \left\{e^{j\frac{2\pi i L}{M}} \mid i = 0,1,\cdots,M-1\right\} &= \left\{e^{j\frac{2\pi i}{M}} \mid i = 0,1,\cdots,M-1\right\}. \end{aligned}$$

Since $e^{j2\pi k} = 1 \forall k \in \mathbb{Z}$, we have $e^{j\frac{2\pi i L}{M}} = e^{j\frac{2\pi (iL \bmod M)}{M}}$. Hence, the equivalent condition is

$$\{(iL) \bmod M \mid i = 0,1,\cdots,M-1\} = \{0,1,\cdots,M-1\}. \qquad (3)$$

Let $0 \le i_1 \le M-1$ and $0 \le i_2 \le M-1$ such that $i_1 \ne i_2$. Without loss of generality, consider $i_1 < i_2$. Using the following identity on modulo operation

$$(a-b) \bmod M = (a \bmod M - b \bmod M) \bmod M,$$

we have,

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M = ((i_1 - i_2) L) \mod M. \quad (4)$$

Case L and M are relatively prime:

Since $0 < i_1 - i_2 < M$, and $\gcd(L, M) = 1$, $((i_1 - i_2) L) \mod M \neq 0$. Therefore from (4),

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M \neq 0,$$
$$\implies (i_1 L) \mod M \neq (i_2 L) \mod M.$$

We have proved that $i_1 \neq i_2 \implies (i_1 L) \mod M \neq (i_2 L) \mod M \, \forall i_1, i_2 \in \{0, 1, 2 \cdots, M-1\}$. Therefore, when $\gcd(L, M) = 1$, equation (3) holds true.

Case M divides L:

Let $L = P \times M$, $P > 1$. Therefore, it is possible to chose $i_1 = i_2 + M$. Under this condition,

$$((i_1 - i_2) L) \mod M = (ML) \mod M = 0.$$

Therefore,

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M = 0$$
$$\implies (i_1 L) \mod M = (i_2 L) \mod M.$$

We have shown that for some choice of $i_1 \neq i_2$, $(i_1 L) \mod M = (i_2 L) \mod M$. Hence, the values $\{(iL) \mod M\}_{i=0}^{M-1}$ are not distinct. Therefore, when M divides L, equation (3) does not hold true.

Case $\gcd(M, L) = G > 1$:

Let $M = G \times P_M$ and $L = G \times P_L$. We can chose $i_1 = i_2 + G$. Under this condition, $e^{j2\pi\frac{iL}{M}} = e^{j2\pi\frac{iP_L}{P_M}}$. Therefore, $\left\{e^{j2\pi\frac{iP_L}{P_M}} \mid i = 0, 1, \cdots, M-1\right\}$ has P_M distinct values. Therefore, equation (3) does not hold true under this condition.

Hence, the equation (3) holds true iff L and M are relatively prime. This proves that M fold decimator and L fold upsampler blocks can be interchanged iff L and M are relatively prime.

(Time domain analysis) From the definitions of decimator and upsampler,

$$x_1[n] = x[Mn].$$
$$y_1[n] = \begin{cases} x_1\left[\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise,} \end{cases}$$
$$y_1[n] = \begin{cases} x\left[M\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Similarly,

$$x_2[n] = \begin{cases} x_1\left[\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases}$$
$$y_1[n] = x_1[Mn],$$
$$y_1[n] = \begin{cases} x\left[\frac{Mn}{L}\right], & Mn \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

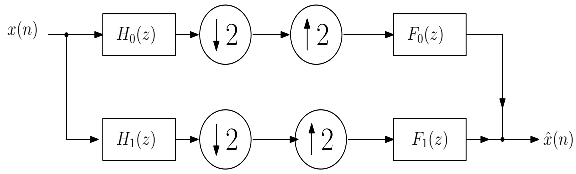
From equations (5) and (6), the outputs are same iff n is a multiple of L when ever Mn is a multiple of L.

Case $\gcd(L, M) = 1$**:** Trivial in this case that L divides $Mn \iff L$ divides n.

Case $\gcd(L, M) = P \neq 1$**:** Let $L = P \times Q$. In this case L divides Mn when ever Q divides n. Hence L divides $Mn \nRightarrow L$ divides n.

Therefore, the outputs are same iff L and M are relatively prime.

PROBLEM 6:
Consider the two channel QMF bank shown below where the analysis filters are given by

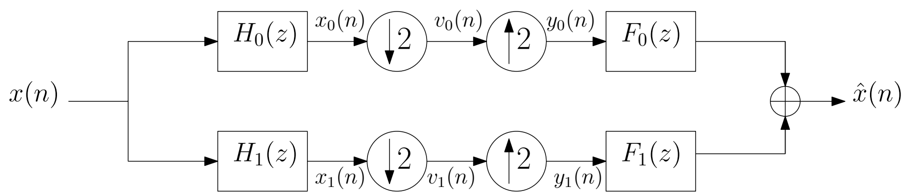


$$H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}; H_1(z) = H_0(-z).$$

Find a set of stable synthesis filters that result in perfect reconstruction. (4 points)

Solution:

Let us label the signals as various junctions.



The signals at various nodes of the figure are

$X_0(z) = H_0(z)X(z)$

$X_1(z) = H_1(z)X(z)$

$Y_0(z) = \frac{X_0(z)+X_0(-z)}{2}$

$Y_1(z) = \frac{X_1(z)+X_1(-z)}{2}$

$\hat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z) = \frac{1}{2}\left[X_0(z)F_0(z) + X_1(z)F_1(z)\right] + \frac{1}{2}\left[X_0(-z)F_0(z) + X_1(-z)F_1(z)\right]$

$\implies \hat{X}(z) = \frac{1}{2}X(z)\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right] + \frac{1}{2}X(-z)\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]$

To force aliasing to zero,

$$[H_0(-z)F_0(z) + H_1(-z)F_1(z)] = 0 \quad \cdots (1)$$

i.e. $\frac{F_0(z)}{F_1(z)} = -\frac{H_1(-z)}{H_0(-z)}$

Then we have $\hat{X}(z) = \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]X(z)$

or $\hat{X}(z) = T(z)X(z)$ where $T(z) = \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]$

For perfect reconstruction $T(z) = cz^{-n_0}$

which implies $\hat{x}(n) = cx(n - n_0)$ and

$$H_0(z)F_0(z) + H_1(z)F_1(z) = 2 \quad \cdots (2)$$

Solving (1) and (2) by substituting $H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}$ and $H_1(z) = H_0(-z)$ we get,

$$F_0(z) = \frac{2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}}{24z^{-1} + 32z^{-3} + 14z^{-5} + 2z^{-7}}$$

$$F_1(z) = \frac{-2 + 6z^{-1} - z^{-2} + 5z^{-3} + z^{-5}}{24z^{-1} + 32z^{-3} + 14z^{-5} + 2z^{-7}}$$