RECONSTRUCTION

Once we have formulated a procedure for signal decomposition, what it really boils down is the goal what we need 'next'

Compression : We may want to identify the "least energy" components and null off the details $\in$ that space i.e., $\{w_k, k \in \mathbb{Z}\}$ & get an approximate signal back.

MPEG 4 + ...

De Noising : We might want to identify spikes at a certain scale 'j' and higher; and notch out such spurious signals. Post this, we can recover the signal.

There are plenty of other applications, such as, in pattern recognition etc where wavelets can be used towards the '**feature extraction**' step

To obtain a reconstruction procedure, let us start with a signal of the form

$$f(t) = f_0(t) + \sum_{i=0}^{j-1} w_i(t) \; ; \quad w_\ell \in W_\ell$$

Here, $$f_0(t) = \sum_{k \in \mathbb{Z}} a_k^{(0)} \phi(t-k) \in V_0$$

$$w_\ell(t) = \sum_{k \in \mathbb{Z}} b_k^{(\ell)} \psi(2^\ell t - k) \in W_\ell$$

$$0 \le \ell \le j-1$$

GOAL: Rewrite $f(t)$ in terms of $\{\phi(2^j t - \ell),\ \ell \in \mathbb{Z}\}$

(INSIGHTS).

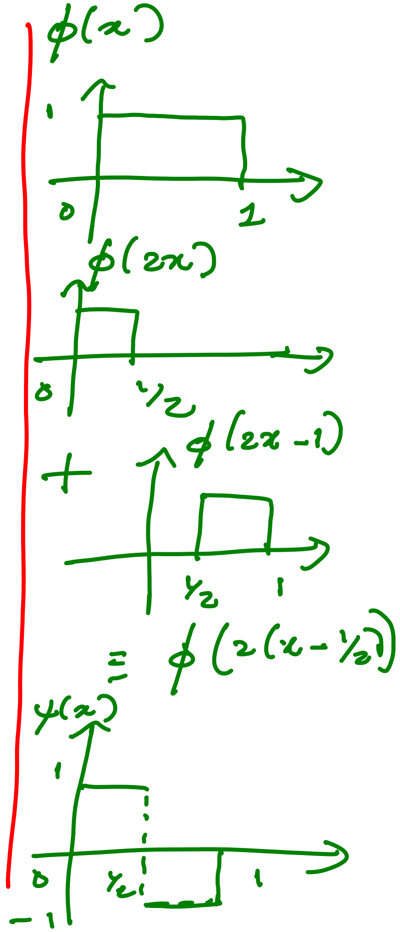
PROCEDURE:

We shall start with our friendly equations

$$\phi(2^{j-1} t) = \phi(2^j t) + \phi(2^j t - 1)$$

$$\psi(2^{j-1} t) = \phi(2^j t) - \phi(2^j t - 1)$$

D



$$\phi(t) = \phi(2t) + \phi(2t-1) \quad \text{———} \quad \mathrm{I.A}$$

$$\psi(t) = \phi(2t) - \phi(2t-1) \quad \text{———} \quad \mathrm{I.B}$$

$\Big\}$ I

We have,

$$f_0(t) = \sum_{k \in Z} a_k^{(0)} \phi(t-k)$$

$$= \sum_{k \in Z} a_k^{(0)} \phi(2t-2k) + \sum_{k \in Z} a_k^{(0)} \phi(2t-2k-1) \quad \text{————} \quad \overline{\underline{\mathrm{II}}}.A$$

So,

$$f_0(t) = \sum_{k \in \mathbb{Z}} \hat{a}_{\ell}^{(1)} \phi(2t - \ell) \quad \text{where}$$

$$\hat{a}_{\ell}^{(1)} = \begin{cases} a_k^{(0)} & \text{if } \ell = 2k \text{ (even)} \\ a_k^{(0)} & \text{if } \ell = 2k+1 \text{ (odd)} \end{cases}$$

|||ly let us consider $w_0(t)$

$$w_0(t) = \sum_{k \in \mathbb{Z}} b_k^{(0)} \psi(t-k)$$

$$w_0(t) = \sum_{k \in \mathbb{Z}} \hat{b}_\ell^{(1)} \phi(2t-\ell) \qquad (\because W_0 \subset V_1)$$

$$\hat{b}_\ell^{(1)} = \begin{cases} b_k^{(0)} & \ell = 2k \\ -b_k^{(0)} & \ell = 2k+1 \end{cases}$$

Adding II.B & II.C,

$$f_0(t) + w_0(t) = \sum_{\ell \in \mathbb{Z}} a_\ell^{(1)} \phi(2t-\ell)$$

$$a_\ell^{(1)} = \begin{cases} a_k^{(0)} + b_k^{(0)} & \text{if } \ell = 2k \\ a_k^{(0)} - b_k^{(0)} & \ell = 2k+1 \end{cases} \qquad \text{II.D}$$

We can continue the process i.e., steps II.A – II.D

with $w_1(t) = \sum_{k \in \mathbb{Z}} b_k^{(1)} \psi(2t-k)$

Let us see how this works!

Replace t by $2t-k$ in I i.e., I.A & I.B,

$$\phi(2t-k) = \phi(2^2t-2k) + \phi(2^2t-2k-1)$$

$$\psi(2t-k) = \phi(2^2t-2k) - \phi(2^2t-2k-1)$$

III

Consider $f_0(t) + w_0(t) + w_1(t)$

$$= \sum_{l\in Z} a_l^{(2)} \phi(2^2t-l)$$

$$a_l^{(2)} = \begin{cases} a_k^{(1)} + b_k^{(1)} & l = 2k \\ a_k^{(1)} - b_k^{(1)} & l = 2k+1 \end{cases}$$

From the above, we are ready to obtain the <u>reconstruction procedure</u> for Haar wavelets

Theorem : Suppose

$f = f_0 + w_0 + w_1 + \dots + w_{j-1}$ with

$f_0(t) = \sum_{k \in Z} a_k^{(0)} \phi(t-k) \in V_0$ and

$w_p(t) = \sum_{k \in Z} b_k^{(p)} \psi(2^p t - k) \in W_p$

for $0 \le p < j$, then

$$f(t) = \sum_{\ell \in \mathbb{Z}} a_\ell^{(j)} \phi(2^j t - \ell) \in V_j$$

where $a_\ell^{(p)}$ can be determined '<u>recursively</u>' for $p = 1, 2, \ldots, j$ using

$$a_\ell^{(p)} = \begin{cases} a_k^{(p-1)} + b_k^{(p-1)} & \ell = 2k \\ a_k^{(p-1)} - b_k^{(p-1)} & \ell = 2k+1 \end{cases}$$

Home Work

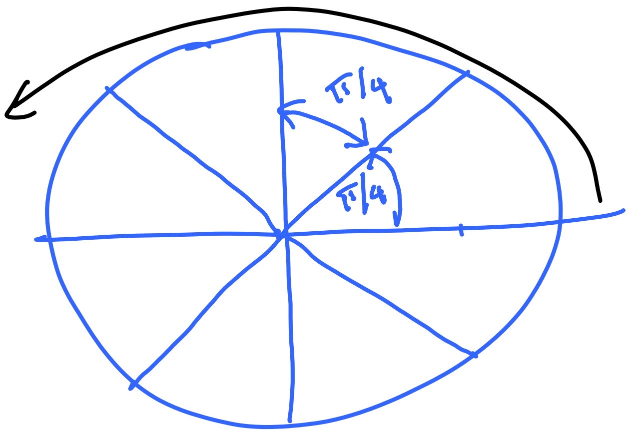
Obtain the Haar decomposition for

$$f(t) = \begin{cases} 3 & 0 \le t < \frac{1}{4} \\ -1 & \frac{1}{4} \le t < \frac{3}{8} \\ 2 & \frac{3}{8} \le t < \frac{5}{8} \\ 0 & 5/8 \le t < 1 \end{cases}$$

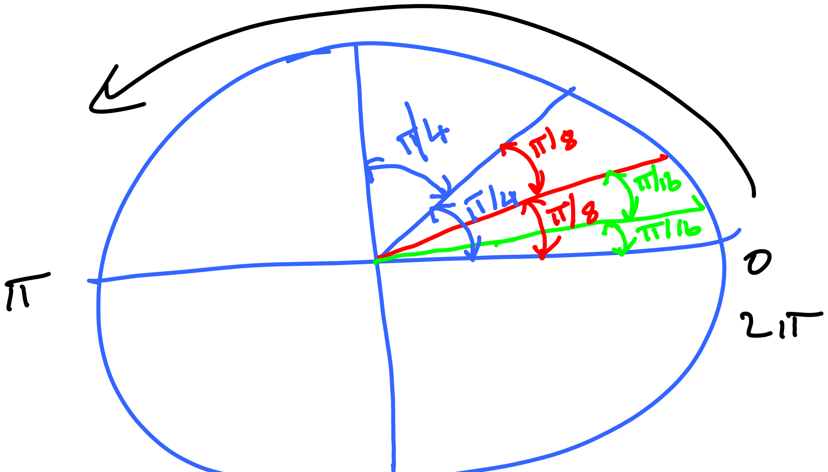
Obtain the signal after notching out any spike of $\frac{1}{8}$ time units. Sketch Fourier spectrum through all the scales

Disk
Partition
Diagram

$e^{j2\pi/N}$



Uniform resolution

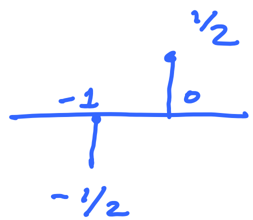


Multiresolution

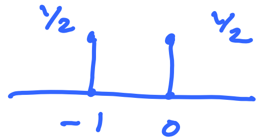
Haar Wavelet Decomposition & link to filter banks

Let us look into wavelet decomposition from a filter bank perspective.

Let $\underline{h} = \left(\underset{\substack{\uparrow \\ -1}}{-\frac{1}{2}} , \underset{\substack{\uparrow \\ 0}}{\frac{1}{2}} \right)$ (HPF)



$\underline{\ell} = \left(\underset{\substack{\uparrow \\ -1}}{\frac{1}{2}} , \underset{\substack{\uparrow \\ 0}}{\frac{1}{2}} \right)$ (LPF)



Let $\{x_k\} \in \ell^2$ i.e., $\ell_2 : \{ x : \|x\|_2 < \infty \}$

$$y_H[k] = \underline{h} * x = \frac{1}{2}x[k] - \frac{1}{2}x[k+1]$$

Convolution operator

$$y_L[k] = \underline{l} * x = \frac{1}{2}x[k] + \frac{1}{2}x[k+1]$$

Keeping only even subscripts / indices via $\downarrow 2$

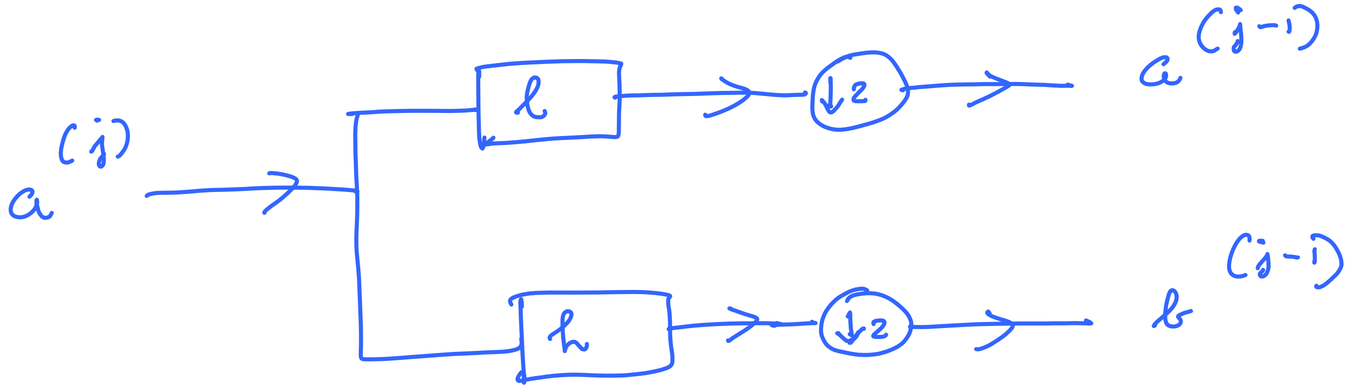
$$y_H[2k] = \frac{1}{2}\left(x_{2k} - x_{2k+1}\right)$$

$$y_L[2k] = \frac{1}{2}\left(x_{2k} + x_{2k+1}\right)$$

Let us apply this idea to the scaling and wavelet coeffts. going from level 'j' to level 'j-1'.

$$b_k^{(j-1)} = \underbrace{y_H(k)}_{a^{(j)} * \underline{h}} \longrightarrow \downarrow 2 \longrightarrow$$

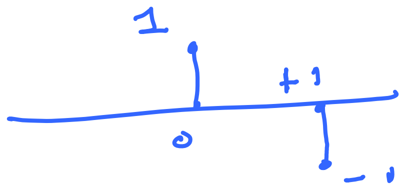
$$a_k^{(j-1)} = \underbrace{y_L(k)}_{a^{(j)} * \underline{\ell}} \longrightarrow \downarrow 2 \longrightarrow$$



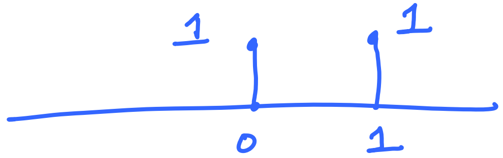
ANALYSIS STAGE

Similarly, the reconstruction procedure involves synthesis filters.

Let $$\underline{\tilde{h}} = \begin{array}{c} \begin{pmatrix} 1 & \quad -1 \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow \\ 0 & \quad 1 \end{matrix} \end{array}$$



$$\underline{\tilde{\ell}} = \begin{array}{c} \begin{pmatrix} 1 & \quad 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow \\ 0 & \quad 1 \end{matrix} \end{array}$$



Following the same idea we did before,

$$\underline{\tilde{h}} * x = x_k - x_{k-1}$$

$$\underline{\tilde{\ell}} * x = x_k + x_{k-1}$$

Suppose $\{x\}$ & $\{y\}$ are sequences with zeros in their odd indices.

$$(\tilde{h} * x)_\ell = \begin{cases} x_{2k} & \ell = 2k \\ -x_{2k} & \ell = 2k+1 \end{cases} \quad (1)$$

$$(\tilde{\ell} * y)_\ell = \begin{cases} y_{2k} & \ell = 2k \\ y_{2k} & \ell = 2k+1 \end{cases} \quad (2)$$

Adding 1 & 2, we get

$$(\tilde{\underline{h}} * x)_\ell + (\tilde{\underline{\ell}} * y)_\ell = \begin{cases} x_{2k} + y_{2k} ; & \ell = 2k \\ -x_{2k} + y_{2k} ; & \ell = 2k+1 \end{cases}$$

Choosing $x_{2k} = b_k^{(j-1)}$ and

$y_{2k} = a_k^{(j-1)}$

we have exactly what we needed from the previous Theorem

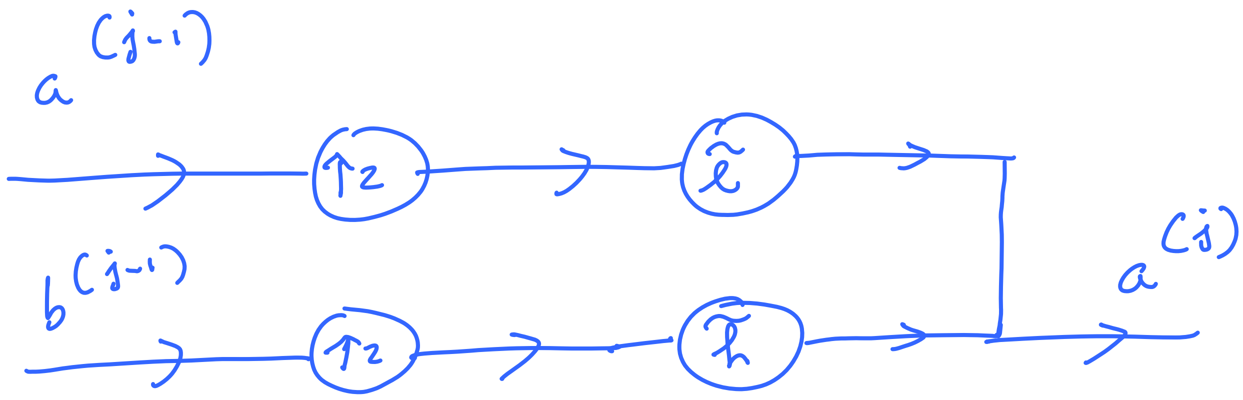
Summarizing Haar reconstruction.

$$a_\ell^{(p)} = \begin{cases} a_k^{(p-1)} + b_k^{(p-1)} ; & \ell = 2k \\ a_k^{(p-1)} - b_k^{(p-1)} ; & \ell = 2k+1 \end{cases}$$

Point to note : Interpret $\{x\}$ & $\{y\}$ as upsampled seq. $(\uparrow 2)$

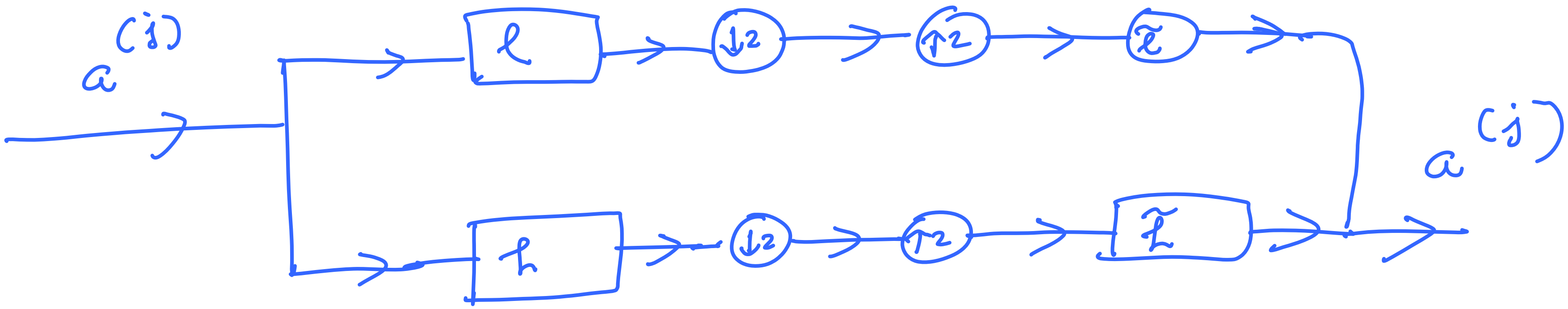
$$\underline{x} = (\dots \; 0 \; b_{-1}^{(j-1)} \; 0 \; b_0^{(j-1)} \; 0 \; \dots)$$

$$\underline{y} = (\dots \; 0 \; a_{-1}^{(j-1)} \; 0 \; a_0^{(j-1)} \; 0 \; \dots)$$

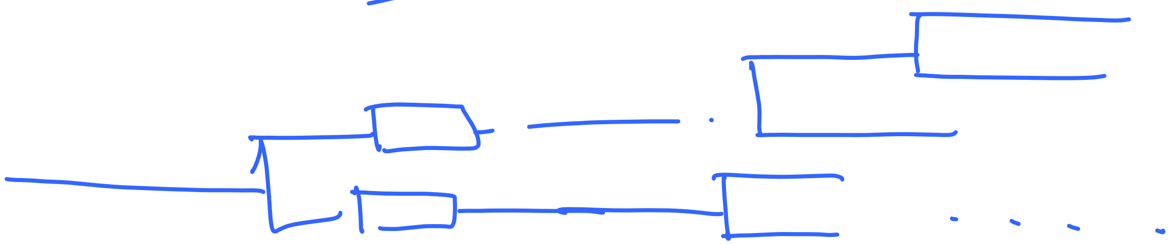


SYNTHESIS STAGE

Now, we can combine both the analysis & synthesis stages



COMBINING ANALYSIS AND SYNTHESIS STAGES



The Idea of Sampling

Let $X(\omega)$ be the spectrum of $x(t)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

If $X(\omega)$ is assumed to be zero outside the band $|\omega| < 2\pi B$,

$$x(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} X(\omega)\, e^{j\omega t}\, d\omega$$

Let $t = \dfrac{n}{2B}$

$$x\left(\frac{n}{2B}\right) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} X(\omega)\, e^{j\omega \frac{n}{2B}}\, d\omega$$

The L.H.S has $x(t)$ at the sampling points.

The integral on the right is essentially the n^{th} coefft in the Fourier series expansion of $X(\omega)$ over the interval $[-B, B]$ as a fundamental period.

$\left\{ x\left(\frac{n}{2B}\right) \right\}$ determine the F. coeffts in the series expansion of $X(\omega)$

Since $X(\omega)$ is zero for frequencies $> B$ &

$X(\omega)$ is determined fully if the coeffts are known,

the samples $\left\{ x\left(\frac{n}{2B}\right) \right\}$ determine $x(t)$ completely.

<u>Problem</u> : How do we '<u>re</u>' construct $x(t)$ from the samples ?

Let us start with the Dirac comb function

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{k=-\infty}^{\infty} c_n e^{j2\pi \frac{k}{T} t}$$

$c_n = \frac{1}{T}$

Periodic $\Rightarrow$ F. series representation

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j2\pi \frac{k}{T} t} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k/T)$$

Consider

$$\sum_{k=-\infty}^{\infty} F(k) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi k t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \underbrace{\sum_{k=-\infty}^{\infty} e^{-j2\pi k t}}_{\sum_{n=-\infty}^{\infty} \delta(t-n)} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, \delta(t-n)\, dt = \sum_{n=-\infty}^{\infty} f(n)$$

|||ly Consider

$$\sum_{k=-\infty}^{\infty} S(\omega + k/T) = \sum_{k=-\infty}^{\infty} \mathcal{F}\left(s(t)\, e^{-j2\pi \frac{k}{T} t}\right)$$

$$= \mathcal{F}\Bigg(s(t) \underbrace{\sum_{k=-\infty}^{\infty} e^{-j2\pi \frac{k}{T} t}}_{T \sum_{n=-\infty}^{\infty} \delta(t-nT)}\Bigg)$$

$$= \mathcal{F}\left(s(t)\, T \sum_{n=-\infty}^{\infty} \delta(t-nT)\right)$$

$$= \mathcal{F}\left(\sum_{n=-\infty}^{\infty} s(nT)\, T\, \delta(t-nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} T \cdot s(nT)\, \mathcal{F}\left(\delta(t-nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} T \cdot s(nT)\, e^{-j2\pi nT\omega}$$

Sampling process converts a continuous time signal into a signal of discrete time.

Sampling Theorem :

If a signal $s(t)$ contains no frequencies outside B Hz, it is completely determined by its values at a sequence of points spaced $> \frac{1}{2B}$ seconds apart.

Let us consider the periodic summation of $S(f)$

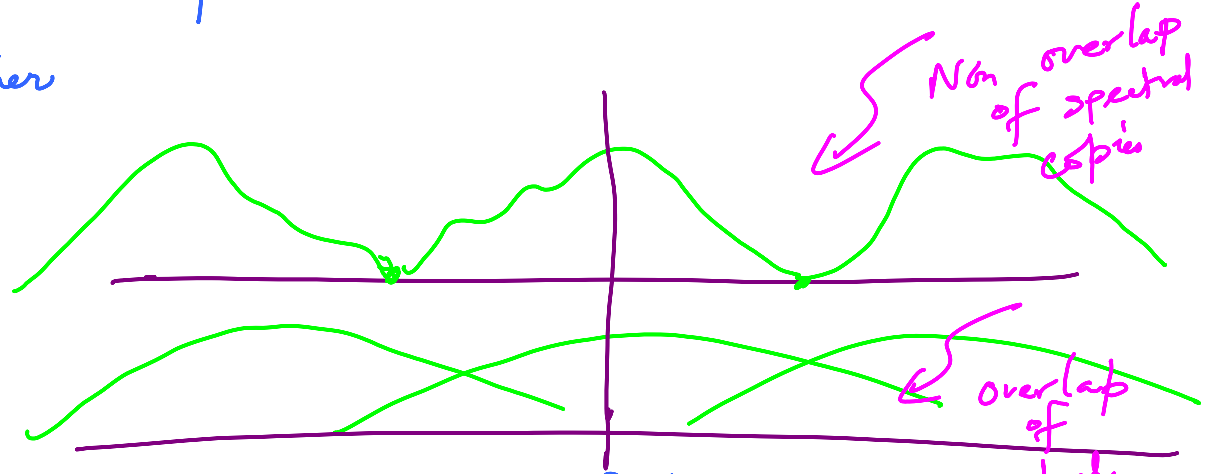
$$S_{\text{periodic sum}}(f) = \sum_{k=-\infty}^{\infty} S(f - k f_s)$$

where $f_s = \frac{1}{T}$ "sampling rate".

$$= \sum_{n=-\infty}^{\infty} T \, s(nT) \, e^{-j2\pi nTf}$$

Copies of $S(f)$ in multiples of f_s, translated are added!

For bandlimited signals i.e., $X(f) = 0;\ |f| \geq B$ & sufficiently large f_s, it is possible for the copies to be distinct from each other



If the <u>Nyquist criterion</u> is not satisfied,

adjacent copies overlap $\Longrightarrow$ aliasing effect

Derive the interpolation formula

$S(f)$ can be used to recover $S_{\text{periodic sum}}(f)$

i.e., with $k = 0$

$$S(f) = H(f)\, S_{\text{periodic sum}}(f)$$

$$H(f) \triangleq \begin{cases} 1 & |f| < B \\ 0 & |f| > f_s - B \end{cases}$$

CRITICAL POINT IS AT $B = f_s/2$ i.e., @ Nyquist

Use the fact

$$H(f) = \mathrm{rect}(f/f_s) = \begin{cases} 1 & |f| < \frac{f_s}{2} \\ 0 & |f| > \frac{f_s}{2} \end{cases}$$

$$S(f) = \mathrm{rect}(f/f_s)\, S_{\text{periodic sum}}(f)$$

$$= \mathrm{rect}(Tf) \sum_{n=-\infty}^{\infty} T\, s(nT)\, e^{-j2\pi nTf}$$

$$= \sum_{n=-\infty}^{\infty} s(nT)\, \underbrace{T \cdot \mathrm{rect}(Tf)\, e^{-j2\pi nTf}}_{\mathcal{F}\left(\mathrm{sinc}\left(\frac{t-nT}{T}\right)\right)}$$

Taking inverse F.T on f.s.

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)\, \underbrace{\text{sinc}\left(\frac{t-nT}{T}\right)}_{\text{Sinc Interpolator}}$$

Other Considerations

A) The sampling theory can be generalized when samples are not taken equally spaced in time.

Henry Landau on non base band, non uniform sampling

B) Recent well developed theory on Compressed sensing

Idea: This allows for full reconstruction with Sub Nyquist Sampling rate for signals that are sparse i.e., Compressible

Low overall bandwidth but freq. locations are unknown rather than everything in one band

Time - Frequency Localization

Consider a finite energy signal i.e., $\int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$

Let us assume that the signal is centered at zero both in time and frequency.

Let us compute the variance in time and frequency by usual 'time' averaging

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} t^2 \, |s(t)|^2 \, dt}{\|s(t)\|_2^2} = \int_{-\infty}^{\infty} t^2 \frac{|s(t)|^2}{\|s(t)\|_2^2} \, dt$$

Like your pdf

Computing the variance in the frequency domain,

$$\sigma_\omega^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \, |S(\omega)|^2 \, d\omega}{\|S(\omega)\|_2^2} = \int_{-\infty}^{\infty} \omega^2 \underbrace{\frac{|S(\omega)|^2}{\|S(\omega)\|_2^2}}_{\text{Like a pdf}} \, d\omega$$

By Parseval's theorem,

$$\|s(t)\|_2^2 = \|S(\omega)\|_2^2 = \|S\|_2^2 \quad (\text{Energy conservation property})$$

Let us consider the following product

$$\sigma_t^2 \sigma_\omega^2 = \frac{\int_{-\infty}^{\infty} t^2 |s(t)|^2 dt \int_{-\infty}^{\infty} \left| \frac{d}{dt} s(t) \right|^2 dt}{\|s\|_2^4}$$

Due to Fourier transform property

$$\frac{d}{dt} s(t) \xrightarrow{\mathcal{F}} \underbrace{j\omega S(\omega)}_{\text{quantity}}$$

Consider $\displaystyle\int_{-\infty}^{\infty} t^2 |s(t)|^2 \, dt = \int_{-\infty}^{\infty} |t\, s(t)|^2 \, dt$

$$= \|t\, s(t)\|_2^2$$

|||ly $\displaystyle\int_{-\infty}^{\infty} \left|\frac{d}{dt} s(t)\right|^2 dt = \left\|\frac{d}{dt} s(t)\right\|_2^2$

Let us apply Cauchy Schwartz inequality

$$|\langle f, g\rangle|^2 \leq \|f\|^2 \, \|g\|^2 \qquad \text{——— (A)}$$

Using (A), we can write

$$\sigma_t^2 \, \sigma_\omega^2 \geq \frac{1}{\|s\|_2^4} \left| \int_{-\infty}^{\infty} t \, s(t) \, \frac{d}{dt} \overline{s}(t) \, dt \right|^2$$

Using the Hermitian Inner Product

$$\geq \frac{1}{\|s\|_2^4} \left| \mathrm{Re} \int_{-\infty}^{\infty} t \, s(t) \, \frac{d}{dt} \overline{s}(t) \, dt \right|^2 \quad \left(\because |\mathrm{Re}(z)| \leq |z| \right)$$

But, $\quad \mathrm{Re}(z) = \frac{1}{2}\left(z + z^*\right)$

Consider $\quad \mathrm{Re} \int_{-\infty}^{\infty} t \, s(t) \frac{d}{dt} \bar{s}(t) \, dt$

$$\Downarrow$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{2} t \left[s(t) \frac{d}{dt} \bar{s}(t) + \bar{s}(t) \frac{d}{dt} s(t) \right]}_{\text{Let us focus on the term within the integral}} dt$$

Term within the integral = $\frac{1}{2}\, t \, \frac{d}{dt} |s(t)|^2$ $\left(\because |s(t)|^2 = s(t)\bar{s}(t) \right)$

$$\sigma_t^2 \sigma_w^2 \geq \frac{1}{4} \cdot \frac{1}{\|s\|_2^4} \left| \int_{-\infty}^{\infty} t \frac{d}{dt} |s(t)|^2 \, dt \right|^2$$

Let us consider $\int_{-\infty}^{\infty} t \frac{d}{dt} |s(t)|^2 \, dt$ and perform integration by parts

ILATE

$$= \underbrace{\cancel{t\,|s(t)|^2 \Big|_{-\infty}^{\infty}}}_{0} - \underbrace{\int_{-\infty}^{\infty} |s(t)|^2\,dt}_{-\|s\|_2^2}$$

$$\sigma_t^2\,\sigma_\omega^2 \geq \frac{1}{4}\cdot\frac{1}{\|s\|_2^4}\left(-\|s\|_2^2\right)^2$$

$$\sigma_t^2\,\sigma_\omega^2 \geq \frac{1}{4}$$

(ORIGINALLY PROVED BY DENNIS GABOR IN 1946
FATHER OF HOLOGRAPHY

Question : When can we achieve equality

i.e., $\sigma_t^2 \sigma_\omega^2 = \frac{1}{4}$ (Set to this lower bound)

From Cauchy Schwartz inequality,

$$t\, s(t) = k \frac{d}{dt} s(t)$$

Let us group the terms and integrate both sides

$$\frac{d\, s(t)}{s(t)} = \frac{t}{k}\, dt$$

Integrating b.s.

$$\ln s(t) = \frac{t^2}{2K} + c$$

$$s(t) = e^{\frac{t^2}{2K} + c}$$

$$= e^{c} \, e^{\frac{t^2}{2K}}$$

$$= a \, e^{t^2/2K}$$

If $s(t)$ is to be a finite energy signal, K must be a $-$ve real no

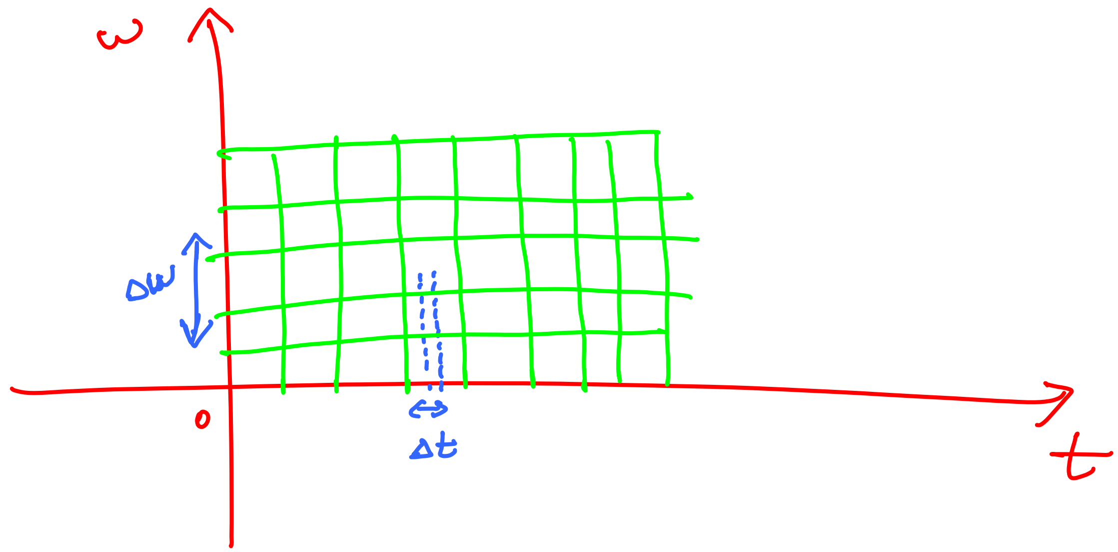
Let $b = -k$

$$s(t) = a\, e^{-\frac{t^2}{2b}}$$

Has the form of a Gaussian pulse

This form led to GABOR TRANSFORMS.

Home Work : Ponder on how the time frequency uncertainty principle applies in the context of wavelet decomposition at different scales



ω
Δω
0
Δt
t

Basic Ideas from Analysis

useful for Signal Processing

Let us discuss "continuity"

Consider S to be a subset of real numbers $\mathbb{R}$ and $f : S \rightarrow R$ is a real valued function defined on S

<u>For example</u> : $S = (0, 2) = \{x \in R : 0 < x < 2\}$

It can be infinite like $S = (0, \infty) = \{x \in R : 0 < x\}$

'Continuous' on S

Definition

The function f is said to be continuous on S iff
$\forall x_0 \in S \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in S$
$$\left[|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \right]$$

Choose $x_0 \in S$. Choose $\varepsilon > 0$. Let $\delta = \delta(x_0, \varepsilon)$

Choose $x \in S$

Assume $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

'UNIFORMLY CONTINUOUS'

Definition

The function f is uniformly continuous on S iff

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \ \forall x_0 \in S \ \forall x \in S$$

$$\Big[\, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \, \Big]$$

Choose $\varepsilon > 0$. Let $\delta = \delta(\varepsilon)$. Choose $x_0 \in S$, $x \in S$

Assume $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

Let us look into a few examples for our understanding.

Example : Let $S = \mathbb{R}$ and $f(x) = mx + c$ $\begin{pmatrix} m > 0 \\ m \neq 0 \end{pmatrix}$

Let us examine if f is uniformly continuous on S

Choose $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{m}$. Choose $x_0 \in \mathbb{R}$

Choose $x \in \mathbb{R}$

Assume $|x - x_0| < \delta$.

Consider $\left| f(x) - f(x_0) \right| = \left| mx + c - (mx_0 + c) \right| = m\,|x - x_0|$

$< m\delta = \varepsilon$

$\therefore$ f is uniformly continuous!

Example: Let us take another example

Suppose $S = \{ x \in \mathbb{R} : 0 < x < 2 \}$ and $f(x) = x^2$

Examine if f is uniformly continuous on S

Steps: Choose $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{4}$

Choose $x_0 \in S$ & $x \in S$

$0 < x_0 < 2$ & $0 < x < 2$

$0 < x_0 + x < 4$. Assume $|x - x_0| < \delta$

Consider $|f(x) - f(x_0)| = |x^2 - x_0^2| = (x + x_0)|x - x_0| < 4|x - x_0|$

$< 4\delta = \varepsilon$

UNIFORMLY CONTINUOUS!

In our examples on $f(x) = mx + c$
and $f(x) = x^2$

$$|f(x_1) - f(x_2)| \leq M |x_1 - x_2| \quad \forall x_1, x_2 \in S$$

Inequality of this form is called
Lipschitz inequality
and the constant M is called the corresponding
Lipschitz constant

Home Work Exercises

1) Examine if $f(x) = x^2$ is uniformly continuous on the $S = (0, \infty)$

2) Examine if $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$

is continuous