Linear Combination of vectors

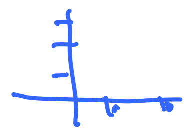
Let S be a V.S. over $\mathbb{R}^m$. Let $p_1, p_2 \dots p_m$ be vectors in S. Then for $c_i \in R$, the linear combination

$$\underline{x} = c_1 \underline{p}_1 + c_2 \underline{p}_2 + \dots + c_m \underline{p}_m \text{ is in } S$$

We can imagine $\{p_i\}_{i=1}^{m}$ are '<u>building blocks</u>' over which other vectors can be obtained.

$(c_1 \ c_2 \ \dots \ c_m)$ can be regarded as the coordinates for $\underline{x}$ over $\{p_i\}_{i=1}^{m}$

$2\hat{i} + 3\hat{j}$



Defn: Let S be a V.S. over $\mathbb{R}^m$. Let $T \subset S$. (possibly ∞ elements)

A point $\underline{x} \in S$ is said to be a linear combination of points in T if $\exists$ a finite set of points $\underline{p}_1\ \underline{p}_2 \ldots \underline{p}_m$ in T & a finite set of scalars $c_1\ c_2 \ldots c_m$ in $\mathbb{R}$ /

$$\underline{x} = c_1 \underline{p}_1 + c_2 \underline{p}_2 + \cdots + c_m \underline{p}_m$$

Examples: Suppose $\underline{p}_1, \underline{p}_2 \in \mathbb{R}^3$.

$\underline{p}_1 = [1\ 0\ 1]^T$ $\quad \underline{p}_2 = [1\ 1\ 0]^T$

$$\underline{x} = c_1 \underline{p}_1 + c_2 \underline{p}_2 = \begin{bmatrix} c_1 + c_2 \\ c_2 \\ c_1 \end{bmatrix};$$

Can $x = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$ be formed from $\underline{p}_1$ & $\underline{p}_2$?

Qns:

1) Is the representation of a vector as a linear combination of other vectors unique?

2) What is the smallest set of vectors that can be used to synthesize any vector in S?

3) Given the set of vectors $\underline{p}_1$ $\underline{p}_2$ ... $\underline{p}_m$, how are the coeffts c_1 c_2 ... c_m found to represent $\underline{x}$?

4) Suppose $\underline{x}$ cannot be exactly represented by $\{\underline{p}_i\}_{i=1}^{m}$, What is the "best" approximation?

Definition : (Linear independence)

Let S be a V.S, and let T be a subset of S. The set T is linearly independent if for each finite non empty subset of T say $\{\underline{p}_1, \underline{p}_2, \dots, \underline{p}_m\}$, the only set of scalars satisfying the equation

$$c_1 \underline{p}_1 + c_2 \underline{p}_2 + \dots + c_m \underline{p}_m = \underline{0}$$

is the "TRIVIAL" solution $c_1 = c_2 = \dots = c_m = 0$

In other words, if we find c_i's that are all not zero such that $\sum_{i=1}^{m} c_i \underline{p}_i = \underline{0}$, $\{\underline{p}_i\}_{i=1}^{m}$ form a linearly dependent set.

Examples :- Suppose $\underline{p}_1 = [2 \quad -3 \quad 4]^T$, $\underline{p}_2 = [-1 \quad 6 \quad 2]^T$, $\underline{p}_3 = [1 \quad 6 \quad 2]^T$. Are they linearly dependent?

$$4\underline{p}_1 + 5\underline{p}_2 + 3\underline{p}_3 = \underline{0}$$

Defn: Let T be a set of vectors in a VS S over a set of scalars R. The set of vectors V that can be reached by all possible (finite) linear combinations of vectors in T is called the 'span' of the vectors.

$$V = \text{span}\{T\}$$

i.e., For any $\underline{x} \in V \ \exists \ \{c_i\} \in R \ / \ \underline{x} = \sum_{i=1}^{m} c_i \underline{p}_i$

Note: If $V = \text{span}(T) \Rightarrow$ is the smallest subspace of S containing T.

Example: Let $\underline{p}_1 = [1\ 1\ 0]^T$ $\underline{p}_2 = [0\ 1\ 0]^T$ in $\mathbb{R}^3$.

$$\underline{x} = c_1 \underline{p}_1 + c_2 \underline{p}_2 = \begin{bmatrix} c_1 \\ c_1 + c_2 \\ 0 \end{bmatrix} \text{ for } c_1, C_2 \in R$$

$V = \text{span}(\underline{p}_1, \underline{p}_2)$ is a subset of the space $\mathbb{R}^3$.

Defn: Let T be a set of vectors in a V.S. S. Let $V \subset S$ be a subspace. If every vector $\underline{x} \in V$ can be written as a linear combination of vectors in T, then T is a '<u>spanning set</u>' of V.

Example: $\underline{p}_1 = [1\ 6\ 5]^T$ $\underline{p}_2 = [-2\ 4\ 2]^T$ $\underline{p}_3 = [1\ 1\ 0]^T$ $\underline{p}_4 = [7\ 5\ 2]^T$ form a **spanning set** of $\mathbb{R}^3$

Verify: $-4\underline{p}_1 + 5\underline{p}_2 - 21\underline{p}_3 + 5\underline{p}_4 = 0$

That $T = \{\underline{p}_1, \underline{p}_2, \underline{p}_3\}$ are linearly independent & span $\mathbb{R}^3$.

Unique Representation Theorem

Theorem : Let S be a vector space and $T \subset S$ and nonempty. The set T is linearly independent **iff** for each nonzero $\underline{x} \in \operatorname{span}(T)$, there is exactly one finite subset of T denoted by $\{\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_n\}$ and a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$$\underline{x} = c_1 \underline{P}_1 + \cdots + c_n \underline{P}_n . \qquad (1)$$

PROOF : Linear independence $\Longrightarrow$ Unique Representation

We shall establish this by Contradiction. Let T be a linearly independent set. Let us assume that $\exists\ \underline{x} \in \text{span}(T)$ whose representation is not unique. Thus, $\exists$ two subsets of T, namely $P = \{ \underline{p}_1, \underline{p}_2, \cdots, \underline{p}_m \}$ and $Q = \{ \underline{q}_1, \underline{q}_2, \cdots, \underline{q}_n \}$ such that

$$\underline{x} = \sum_{i=1}^{m} c_i \underline{p}_i = \sum_{i=1}^{n} d_i \underline{q}_i$$

where c_i's and d_i's are non zero.

Let us rearrange the terms in the representation for $\underline{x}$.

$$\sum_{i=1}^{m} c_i \underline{p}_i - \sum_{i=1}^{n} d_i \underline{q}_i = \underline{0} \quad \text{——} \quad (2)$$

As $\underline{p}_i$s and $\underline{q}_i$s belong to T, if $P \cap Q = \phi$ then $\underline{p}_i$s and $\underline{q}_i$s are different. This contradicts the fact that T is a linearly independent set as their non trivial linear combination cannot sum to zero. Hence, there must be some overlap between the two sets.

Let $m < n$.

Eqn (2) holds only if for every $\underline{p}_i$ $\exists$ some $\underline{q}_j$ such that $\underline{p}_i = \underline{q}_j$ and $c_i - d_j = 0$. This is true as only trivial linear combination of vectors in T can be $\underline{0}$.

Renumbering the elements in Q, we obtain

$$\underline{p}_i = \underline{q}_i \text{ and } c_i = d_i \qquad (3)$$

Thus, $P \subset Q$. From (2) and (3),

$$\sum_{i=m+1}^{n} d_i \underline{q}_i = \underline{0} \qquad (4)$$

As, if $\underline{q}_i$ s are non zero, they should be linearly independent and d_i are non zero, the only possible solution is $\underline{q}_i = \underline{0}$.

Neglecting the zero vector, we define

$$Q = \{\underline{q}_1, \ldots, \underline{q}_n\} = P$$

$\Rightarrow$ The representation is <u>unique</u>

Converse: Unique Representation $\Longrightarrow$ Linear Independence

We shall establish this via Contradiction.

Let every vector $\underline{x} \in \text{span}(T)$ have a unique representation in terms of vectors in $T = \{\underline{t}_1, \dots \underline{t}_k\}$. Let us assume that T is a linearly dependent set, then $\exists$ $a_1, a_2, \dots, a_k$ where atleast one a_i is non zero /

$$\sum_{i=1}^{k} a_i \underline{t}_i = \underline{0} \quad (5)$$

Let a_1 be non zero. Consider $\underline{x} = \underline{t}_1 = -\frac{1}{a_1} \sum_{i=2}^{k} a_i \underline{t}_i$ (6)

As $\underline{x}$ does not have a unique representation $\Longrightarrow$ a contradiction $\Longrightarrow$ T is a linearly independent set ▨

Basis

Defn: Let S be a V.S. Let T be a set of vectors from S such that $\text{span}(T) = S$. If T is linearly independent, T is said to the 'Hamel basis" of S.

Examples:

1) $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A natural basis in $\mathbb{R}^3$.

2) $\underline{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{p}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$.

Theorem: If T_1 and T_2 are Hamel bases for a V.S. S, then T_1 and T_2 have the same cardinality.

Proof: Suppose $T_1 = \{\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m\}$ and $T_2 = \{\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_n\}$ be two Hamel bases of S.

Express the point $\underline{q}_1 \in T_2$ as

$$\underline{q}_1 = c_1 \underline{p}_1 + c_2 \underline{p}_2 + \cdots + c_m \underline{p}_m .$$

At least one of c_i's must be non zero. Let that be c_1

$$\underline{p}_1 = \frac{1}{c_1}\left(\underline{q}_1 - c_2 \underline{p}_2 - c_3 \underline{p}_3 - \cdots - c_m p_m\right) \quad c_1 \neq 0$$

$\Rightarrow$ We can eliminate $\underline{p}_1$ as a basis in T_1 & use the set $\{\underline{q}_1, \underline{p}_2, \dots, \underline{p}_m\}$ as a basis set.

Similarly, $d_2 \neq 0$

$$\underline{q}_2 = d_1 \underline{q}_1 + d_2 \underline{p}_2 + \dots + d_m \underline{p}_m$$

$$\underline{p}_2 = \frac{1}{d_2}\left[\underline{q}_2 - d_1 \underline{q}_1 - d_3 \underline{p}_3 - \dots - d_m \underline{p}_m \right]$$

We can eliminate $\underline{p}_2$ from the list so that $\{\underline{q}_1, \underline{q}_2, \underline{p}_3, \dots, \underline{p}_m\}$ to form a basis set.

Doing this iteratively, we get $\{\underline{q}_1, \underline{q}_2, \dots, \underline{q}_m\}$ spanning the same space as $\{\underline{p}_1, \underline{p}_2, \dots, \underline{p}_m\}$. We can conclude that $m \geq n$

Suppose to the contrary $n > m$, then a vector such as $\underline{q}_{m+1}$ which does not fall in $\{\underline{q}_1, \cdots \underline{q}_m\}$ would be linearly dependent with the set $\{\underline{q}_1, \cdots \underline{q}_m\}$ violating that T_2 is a basis.

Do the reverse argument with eliminating $\{q_i\}$ & conclude $n \geq m$.

$\Rightarrow \quad m = n$

$\boxtimes$

Norms & Normed V.S

The mathematical concept of associating with the length of a vector is the "norm". This concept is useful later when we deal Inner Products.

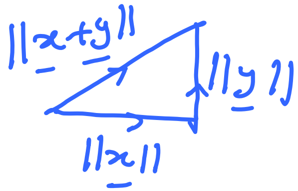
Defn: Let S be a V.S. with elements $\underline{x}$. A real valued function $\|\underline{x}\|$ is said to be the norm of $\underline{x}$ if the foll. hold:

a) $\|\underline{x}\| \geq 0$ for any $\underline{x} \in S$.

b) $\|\underline{x}\| = 0$ iff $\underline{x} = \underline{0}$

c) $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$ α is any scalar

d) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ (Triangle inequality)



Various definitions / metrics

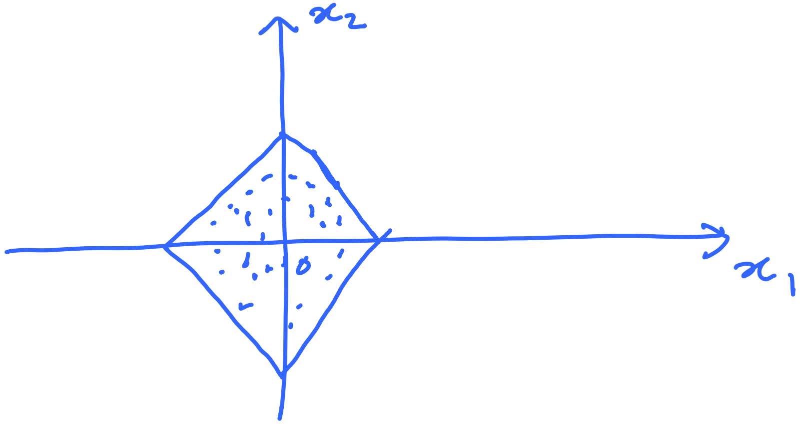
1) L_1 norm : $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$

2) L_p norm : $\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$

3) L_∞ norm : $\|\underline{x}\|_\infty = \max_{i=1,2,\dots,n} |x_i|$

Shapes in $\mathbb{R}^2$

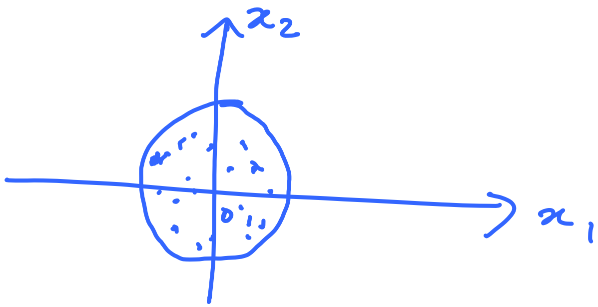
L_1 :



$|x_1| + |x_2| \le 1$

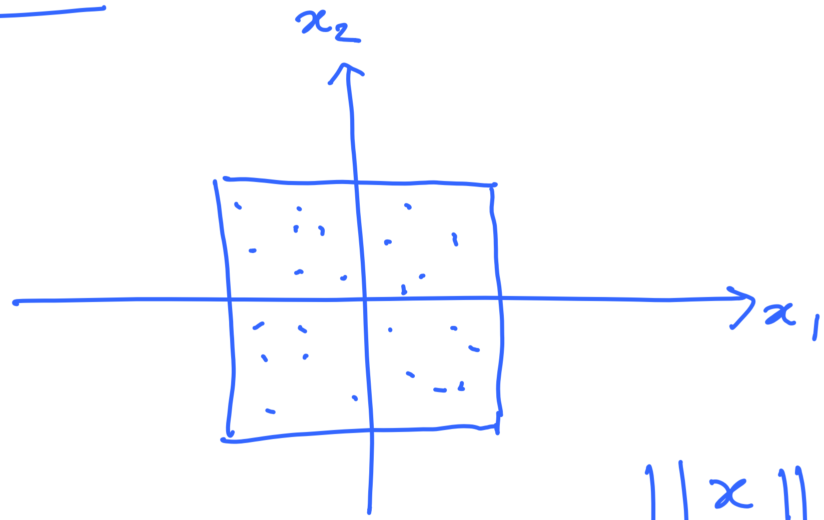
$\|\underline{x}\|_1 \le 1$

L_2 :



$\|x\|_2 \le 1$ $\quad \sqrt{x_1^2 + x_2^2} \le 1$

L_∞ norm



$$\|\underline{x}\|_\infty \le 1$$

$\|\cdot\|$ ly for functions defined over $[a, b]$

L_1 : $\quad \|x(t)\|_1 = \int_a^b |x(t)| \, dt$

L_p : $\quad \|x(t)\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}} \quad 1 \le p < \infty$

L_∞ : $\quad \|x(t)\|_\infty = \sup_{t \in [a, b]} |x(t)|$

Defn:

A vector $\underline{x}$ is normalized if $\|\underline{x}\| = 1$.

It is possible to normalize any vector except $\underline{0}$.

$$\frac{\underline{x}}{\|\underline{x}\|}$$ is the unit vector

Inner Product Spaces

Defn : Let S be a V.S. defined over a scalar field $\mathbb{R}$.
An inner product function $\langle \cdot , \cdot \rangle : S \times S \longrightarrow \mathbb{R}$ with the foll. properties

IP1 : $\langle \underline{x}, \underline{y} \rangle = \overline{\langle \underline{y}, \underline{x} \rangle}$ (cmplx) ; $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$

IP2 : $\langle \alpha \underline{x}, \underline{y} \rangle = \alpha \langle \underline{x}, \underline{y} \rangle \quad \forall \alpha \in \mathbb{R}$

IP3 : $\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$

IP4 : $\langle \underline{x}, \underline{x} \rangle > 0 \quad \forall \underline{x} \neq 0$ & 0 iff $\underline{x} = \underline{0}$.

$$\langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y} = \underline{y}^T \underline{x}$$

$$\underline{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

$$\underline{y} = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}$$

$$i\,\hat{i} + 2i\,\hat{j}$$

$$2-i\,\hat{i} + 3/i\,\hat{j}$$

Inner Products and Inner Product Spaces

Definition : Let S be a V. S. defined over a scalar field $\mathbb{R}$.

An inner product function : $S \times S \longrightarrow \mathbb{R}$ with the foll. properties.

IP 1 : $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$.

IP 2 : $\langle \alpha \underline{x}, \underline{y} \rangle = \alpha \langle \underline{x}, \underline{y} \rangle$.

IP 3 : $\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$.

IP 4 : $\langle \underline{x}, \underline{x} \rangle > 0 \quad \forall \underline{x} \neq \underline{0}$ & 0 iff $\underline{x} = \underline{0}$.

Induced Norm :

In L_2 for $\underline{x} \in \mathbb{R}^n$

$$\langle \underline{x}, \underline{x} \rangle^{\frac{1}{2}} = \|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\underline{x} = (x_1, x_2 \dots x_n)$$

|||ly for functions,

$$\|x(t)\|_2 = \left(\int_a^b |x(t)|^2 \, dt \right)^{\frac{1}{2}}$$

$$\|\underline{x}-\underline{y}\|^2 = \langle \underline{x}-\underline{y}, \underline{x}-\underline{y}\rangle$$
$$= \langle \underline{x}, \underline{x}\rangle - 2\langle \underline{x}, \underline{y}\rangle + \langle \underline{y}, \underline{y}\rangle$$

<u>Theorem</u> : In an inner product space S with induced norm $\|\cdot\|$

$$\langle \underline{x}, \underline{y}\rangle^2 \leq \|\underline{x}\|^2 \|\underline{y}\|^2$$

<u>Proof</u> : Let $\underline{x}$ and $\underline{y}$ be any two vectors in S.

Let us choose an $\alpha \in \mathbb{R}$ given by $\alpha = \dfrac{\langle \underline{x}, \underline{y}\rangle}{\|\underline{y}\|^2}$

$$0 \leq \|\underline{x} - \alpha\underline{y}\|^2$$
$$= \langle \underline{x}, \underline{x}\rangle - 2\alpha\langle \underline{x}, \underline{y}\rangle + \alpha^2\langle \underline{y}, \underline{y}\rangle$$
$$= \langle \underline{x}, \underline{x}\rangle - 2\frac{\langle \underline{x}, \underline{y}\rangle}{\|\underline{y}\|^2}\langle \underline{x}, \underline{y}\rangle + \frac{\langle \underline{x}, \underline{y}\rangle^2}{\|\underline{y}\|^4}\underbrace{\langle \underline{y}, \underline{y}\rangle}_{\|\underline{y}\|^2}$$

$$0 \leq \|\underline{x}\|^2 - 2\frac{\langle \underline{x}, \underline{y}\rangle^2}{\|\underline{y}\|^2} + \frac{\langle \underline{x}, \underline{y}\rangle^2}{\|\underline{y}\|^2}$$

$$0 \leq \|\underline{x}\|^2 - \frac{\langle \underline{x}, \underline{y}\rangle^2}{\|\underline{y}\|^2}$$

$$\Rightarrow \langle \underline{x}, \underline{y}\rangle^2 \leq \|\underline{x}\|^2 \cdot \|\underline{y}\|^2 \qquad \boxtimes$$

CAUCHY SCHWARTZ INEQUALITY.

Exercises : To ponder upon

1) Can we use the Cauchy Schwartz inequality to show that the |Corr. coefft| for jointly distributed random vars. is ≤ 1

2) Define the inner product function from $S \times S \to \mathbb{C}$ & derive the C.S. inequality for this case.

Hint :

$$\underline{v}_1 = i\,\hat{e}_1 + (2-3i)\,\hat{e}_2 \qquad i = \sqrt{-1}$$

$$\underline{v}_2 = 2i\,\hat{e}_1 + (4-6i)\,\hat{e}_2$$

For functions,

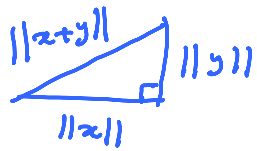
$$\left(\int_a^b f(t)\, g(t)\, dt \right)^2 \leq \int_a^b f^2(t)\, dt \int_a^b g^2(t)\, dt$$

With this, we can consider $\langle \underline{x}, \underline{y} \rangle$ as ~~a~~ measure of some relation between $\underline{x}$, $\underline{y}$

$$\| \underline{x} + \underline{y} \|^2 = \langle \underline{x}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle + 2 \langle \underline{x}, \underline{y} \rangle$$

When can $\langle \underline{x}, \underline{y} \rangle = 0$?

$$\| \underline{x} + \underline{y} \|^2 = \| \underline{x} \|^2 + \| \underline{y} \|^2$$



Recall: $\| \underline{x} \|^2 = \langle \underline{x}, \underline{x} \rangle$

$\| \underline{y} \|^2 = \langle \underline{y}, \underline{y} \rangle$

Angle between vectors

$$\cos\theta = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\|_2 \|\underline{y}\|_2}$$

induced norm

Since $|\cos\theta| \leq 1$,

$$-1 \leq \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\|_2 \|\underline{y}\|_2} \leq 1$$

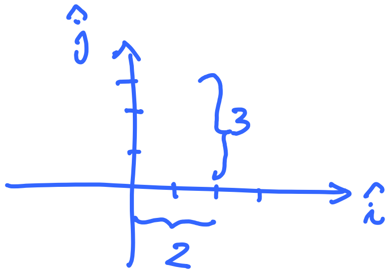
$\langle \underline{x}, \underline{y} \rangle = 0 \implies \underline{x}, \underline{y}$ are 'orthogonal'

Orthonormal

A set of vectors $\{\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_m\}$ are 'orthonormal' if $\langle \underline{p}_i, \underline{p}_j \rangle = \delta_{i,j}$ for pairs i & j

This notion is useful to get a sense of length along the bases

$$2\hat{i} + 3\hat{j}$$



$$\langle \hat{i}, \hat{j} \rangle = 0$$
$$\langle \hat{i}, \hat{i} \rangle = 1$$
$$\langle \hat{j}, \hat{j} \rangle = 1$$

Exercise : Examine if $P_0(t) = 1$ & $P_1(t) = t$ are orthogonal over $[-1, 1]$.