Polyphase Representation of a 2-channel F.B.

Suppose $H_0(z) = E_0(z^2) + z^{-1} E_1(z^2)$

If we assume quadrature mirror property,

$H_1(z) = H_0(-z)$

$H_1(z) = E_0(z^2) - z^{-1} E_1(z^2)$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1} E_1(z^2) \end{bmatrix}$$

(Analysis Bank 2 ϕ representation)

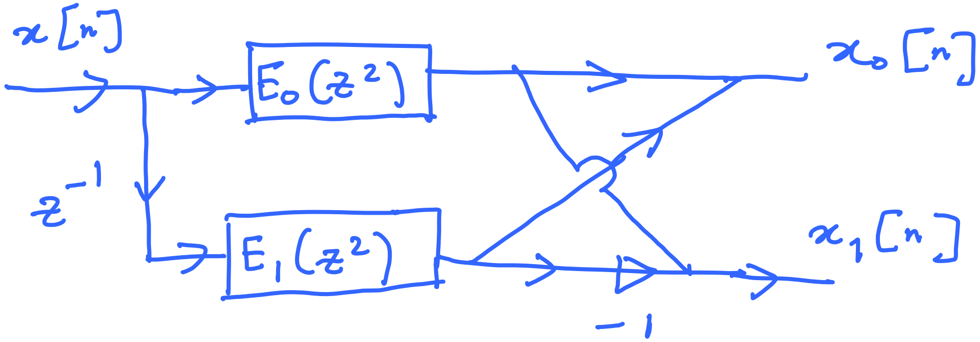
Using alias cancelation conditions,

$$F_0(z) = H_1(-z) \quad ; \quad F_1(z) = -H_0(-z)$$

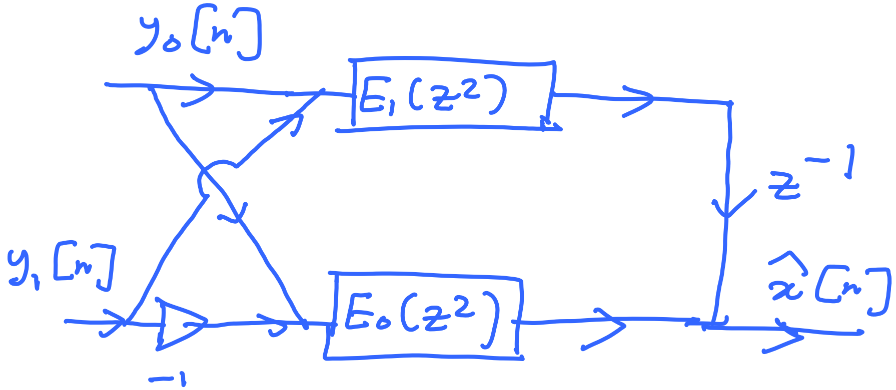
$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(Synthesis filters realized using 2 ϕ representation)

Signal Flow Graphs / Representations for poly ϕ realization



Analysis Bank

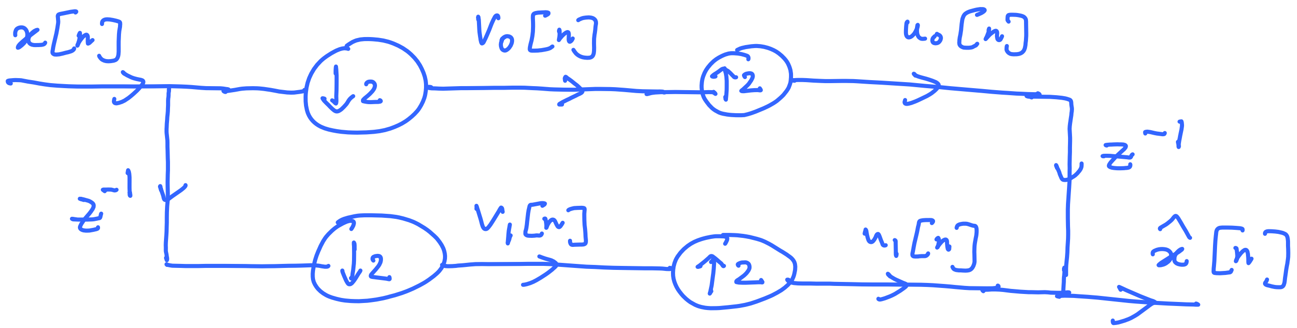


Synthesis Bank

Example : 2 - channel perfect reconstruction system

Suppose $H_0(z) = 1 \qquad H_1(z) = z^{-1}$

$F_0(z) = z^{-1} \qquad F_1(z) = 1$



GOAL: Examine if the system in this example is a P. R. system

Suppose

$$x[n] = \{ \underset{\uparrow n=0}{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \ldots \}$$

$$v_0[n] = \{ 1 \quad 3 \quad 5 \quad 7 \quad \ldots \}$$

$$v_1[n] = \{ x \quad 2 \quad 4 \quad 6 \quad 8 \quad \ldots \}$$

$\underline{x}$: dummy

$$u_0[n] = \{ 1 \quad \underline{0} \quad 3 \quad \underline{0} \quad 5 \quad \underline{0} \quad 7 \quad \ldots \}$$

NOTE:

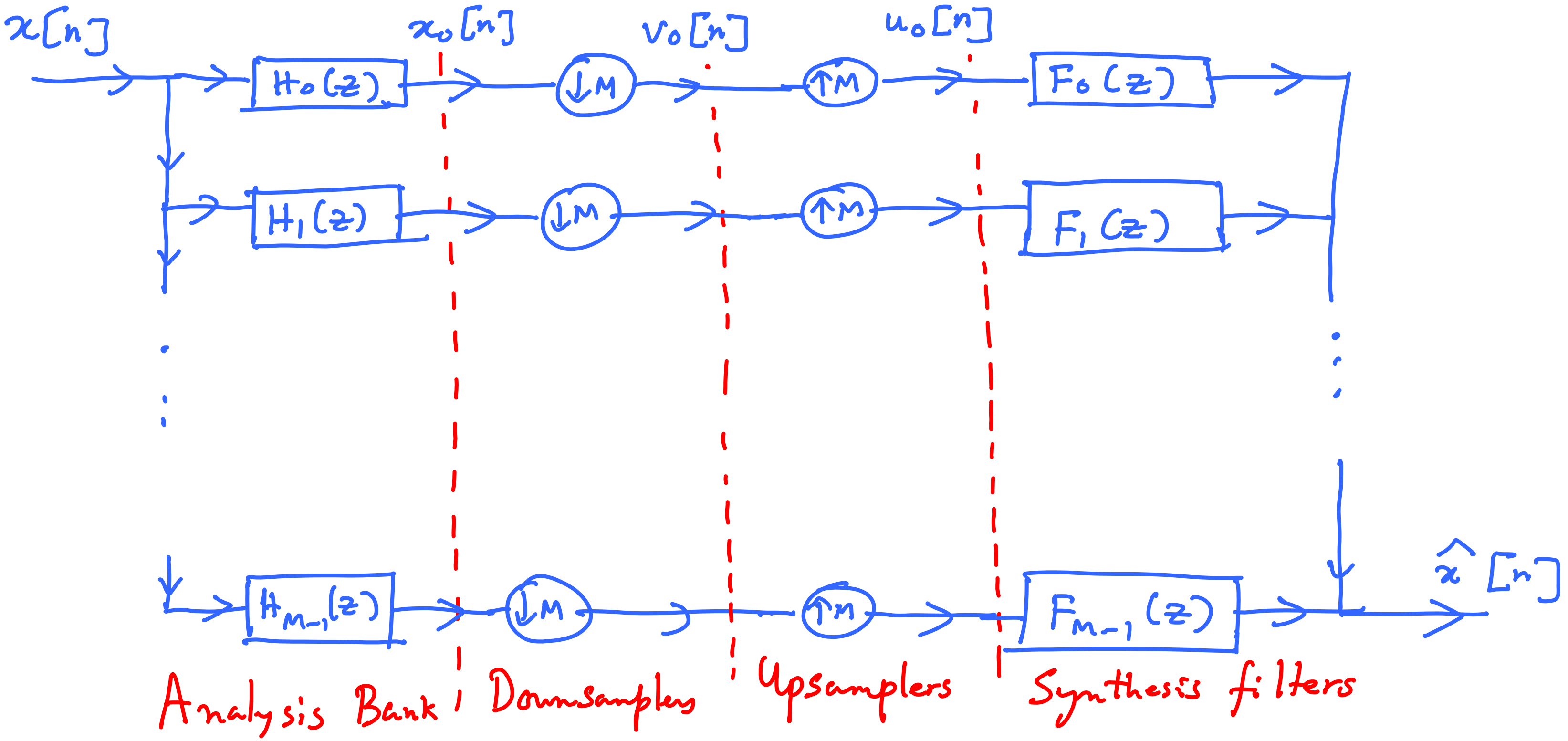
We have P.R. with a delay of 1 unit i.e., z^{-1}

$$u_1[n] = \{ x \quad \underline{0} \quad 2 \quad \underline{0} \quad 4 \quad \underline{0} \quad \ldots \}$$

$$u_0[n-1] = \{ x \quad 1 \quad \underline{0} \quad 3 \quad \underline{0} \quad 5 \quad \underline{0} \quad 7 \quad \ldots \}$$

$$u_1[n] + u_0[n-1] = \{ \underbrace{x}_{n=0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \ldots \}$$

M-channel Filter Banks (GENERALIZATION)



Let $\underline{h}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}$ Analysis Bank

$\underline{f}(z) = \begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}$ transposed synthesis filters

$$\underline{e}(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

delay chain needed for poly phase representation

$$X_k(z) = H_k(z)\, X(z) \qquad k = 0, 1, \cdots, M-1$$

$$V_k(z) = \frac{1}{M} \sum_{l=0}^{M-1} X\left(z^{\frac{1}{M}} w^l\right) H_k\left(z^{\frac{1}{M}} w^l\right) \qquad w = e^{-j\frac{2\pi}{M}}$$

$$U_k(z) = V_k(z^M) = \frac{1}{M} \sum_{l=0}^{M-1} X(z w^l)\, H_k(z w^l)$$

$$\hat{X}(z) = \sum_{k=0}^{M-1} F_k(z)\, U_k(z) = \frac{1}{M} \sum_{l=0}^{M-1} X(z w^l) \sum_{k=0}^{M-1} H_k(z w^l)\, F_k(z)$$

Rewriting in an easier way,

$$\hat{X}(z) = \sum_{l=0}^{M-1} A_l(z)\, X(z w^l) \qquad 0 \le l \le M-1$$

where $$A_l(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z w^l)\, F_k(z)$$

With $z = e^{j\omega}$

$$X(e^{j\omega} w^l) = X\left(e^{j\left(\omega - \frac{2\pi l}{M}\right)}\right)$$

$X(z w^l)$ for $1 \le l \le M-1$ are "ALIAS COMPONENTS"

To avoid aliasing $A_l(z) = 0 \qquad 1 \le l \le M-1$

Let $\underline{A}(z) = \begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix}$

Let

$$\mathbf{H}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zw) & H_1(zw) & \cdots & H_{M-1}(zw) \\ \vdots & & \ddots & \\ H_0(zw^{M-1}) & H_1(zw^{M-1}) & \cdots & H_{M-1}(zw^{M-1}) \end{bmatrix}_{M\times M}$$

Alias component matrix

With aliasing canceled out, the distortion function

$$T(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z) F_k(z) = A_0(z)$$

Now

$$M \begin{bmatrix} A_0(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix} = \mathbf{H}(z)\, \underline{f}(z) = \underline{t}(z)$$

But $A_\ell(z) = 0$ for $1 \leq \ell \leq M-1$

$$\underline{t}(z) = M \begin{bmatrix} A_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = M \begin{bmatrix} T(z) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\hat{X}(z) = A^T(z)\, \underline{X}(z)$$

$$= \frac{1}{M}\, \underline{f}^T(z)\, H^T(z)\, X(z)$$

where

$$\underline{X}(z) = \begin{bmatrix} X(z) \\ X(zw) \\ \vdots \\ X(zw^{(M-1)}) \end{bmatrix}$$

Now,

$$\underline{f}(z) = H^{-1}(z)\, t(z)$$

(Realizable if $H^{-1}(z)$ exists)

For P. R.,

$$\underline{t}(z) = c \begin{bmatrix} z^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Difficulties with AC inversion matrix

a) Unless $|H(z)|$ is $\neq 0$ for every z, we are in trouble towards getting $H^{-1}(z)$

b) $$\underline{f}(z) = \underbrace{\frac{1}{\det(\mathbf{H}(z))}} \operatorname{adj}(\mathbf{H}(z))\, \underline{t}(z)$$

Can give rise to a denominator poly. which may not be just a 'delay' with a 'gain'.

Even if analysis filters are FIR, Synthesis filters can be IIR

$\Longrightarrow$ Stability can be a 'problem'!

c) If $\det(\mathbb{H}(z))$ has zeros on the unit circle i.e., $z = e^{j\omega_0}$

transfer fn. for unaliased part

then $|T(e^{j\omega_0})| = 0$

$\Longrightarrow$ Severe amplitude distortion around ω_0 !

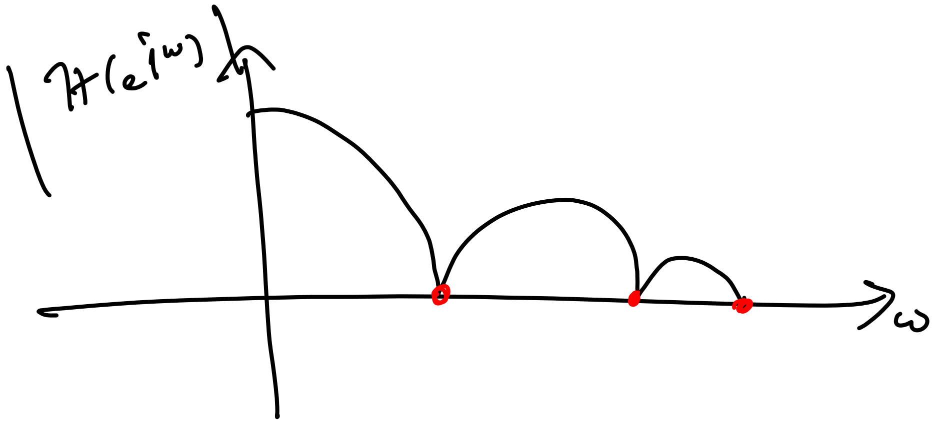
Singularity of $\mathbf{H}(e^{j\omega})$ vs. amplitude distortion

If $T(z)$ has a zero @ $z = e^{j\omega_0}$, then

$\underline{t}(e^{j\omega_0}) = 0$

$\Rightarrow \quad \mathbf{H}(e^{j\omega_0})\,\underline{f}(e^{j\omega_0}) = 0$

$\mathbf{H}(e^{j\omega_0})$ is '<u>singular</u>' unless all synthesis filters $F_k(z)$ have a zero @ ω_0



$|H(e^{j\omega})|$
ω

Polyphase Representation for M-channel Filter Banks

From our previous discussion, by polyphase representation,

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M) \quad \text{(Type 1 representation for analysis filters)}$$

We may write

$$\underbrace{\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\underline{h}(z)} = \underbrace{\begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ \vdots & \ddots & & \vdots \\ E_{M-1,0}(z^M) & \cdots & & E_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbf{E}(z^M)} \underbrace{\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\underline{e}(z)}$$

$$\underline{h}(z) = \mathbf{E}(z^M)\,\underline{e}(z)$$

Ill^{ly}, we can do this for the synthesis filters

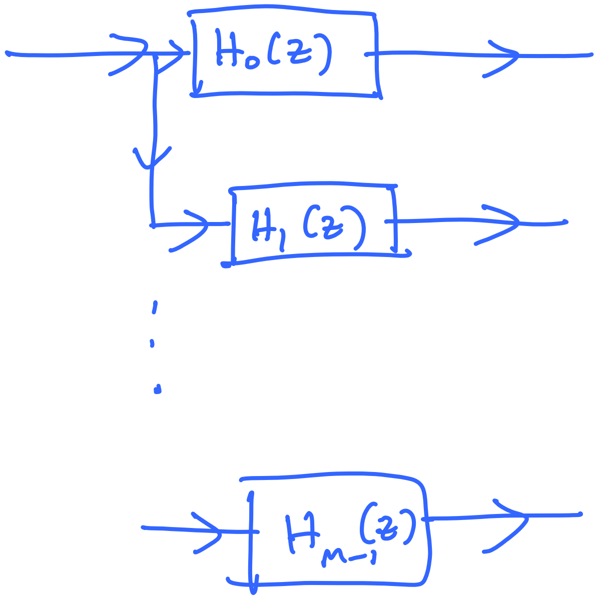
$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{lk}(z^M)$$

(Using Type 2 polyphase representation)

$$\underbrace{\begin{bmatrix} F_0(z) & \cdots & F_{M-1}(z) \end{bmatrix}}_{f^T(z)} = z^{-(M-1)} \cdot \underbrace{\begin{bmatrix} z^{-(M-1)} & \cdots & 1 \end{bmatrix}}_{\tilde{e}(z)} \underbrace{\begin{bmatrix} R_{00}(z^M) & \cdots & R_{0,M-1}(z^M) \\ \vdots & \ddots & \\ R_{M-1,0}(z^M) & & R_{M-1,M-1}(z^M) \end{bmatrix}}_{R(z^M)}$$

$$\underline{f}^T(z) = z^{-(M-1)} \underline{\tilde{e}}(z) R(z^M)$$

$$\tilde{e}(z) = e^T(z)$$



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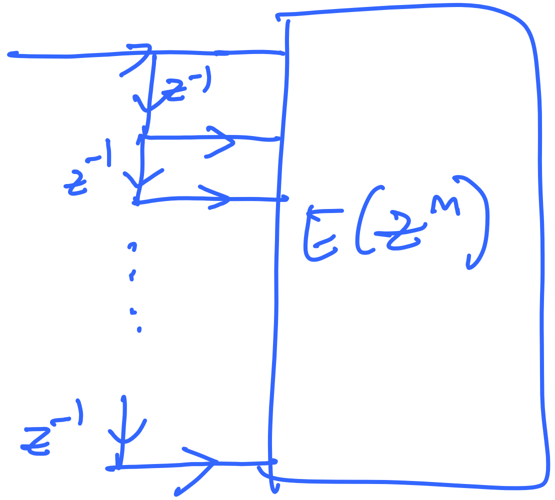
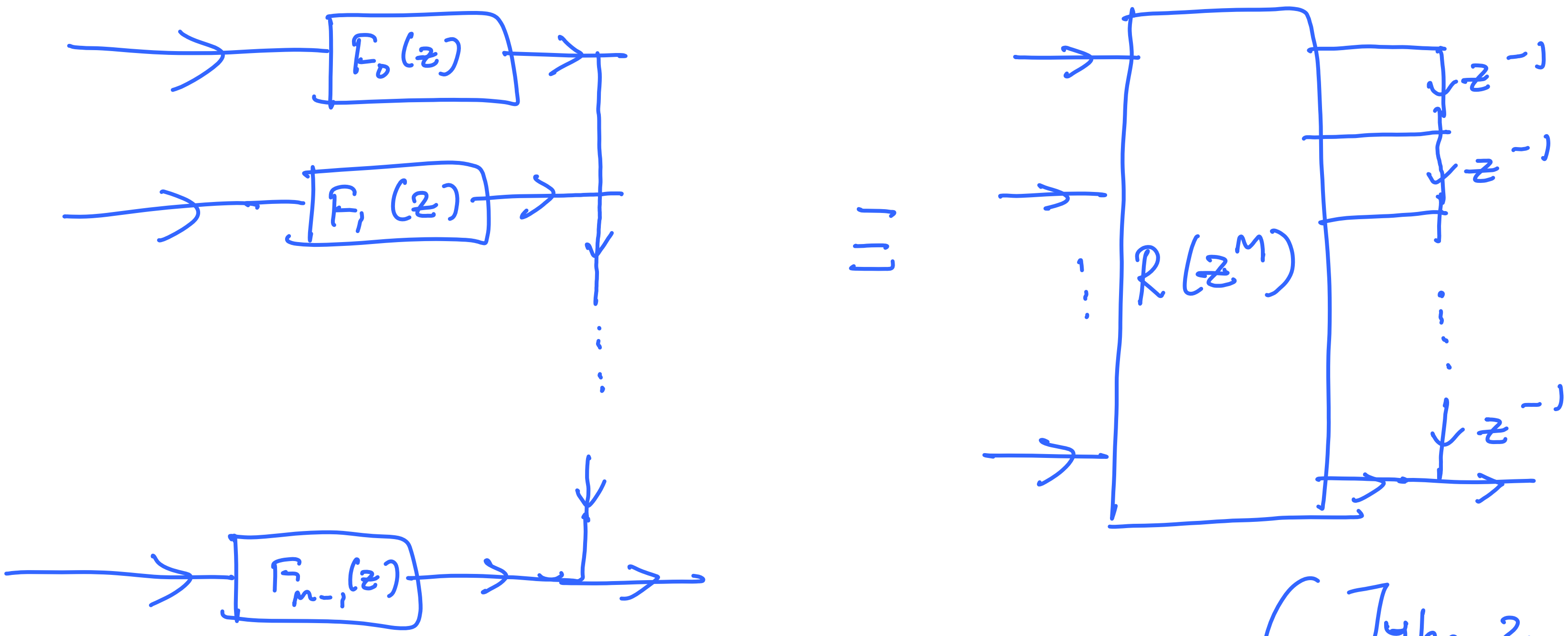
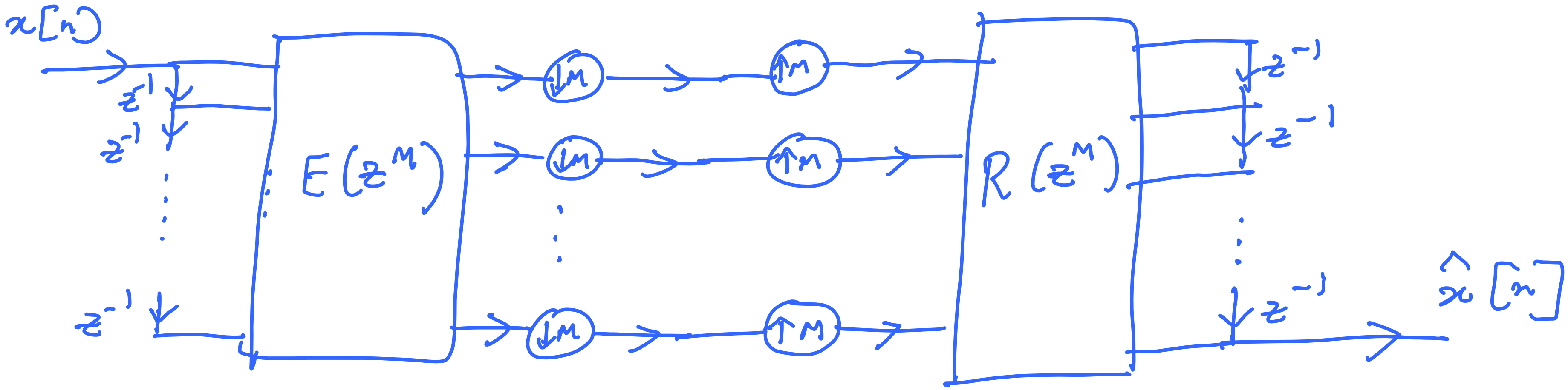


Analysis Bank



Synthesis Bank

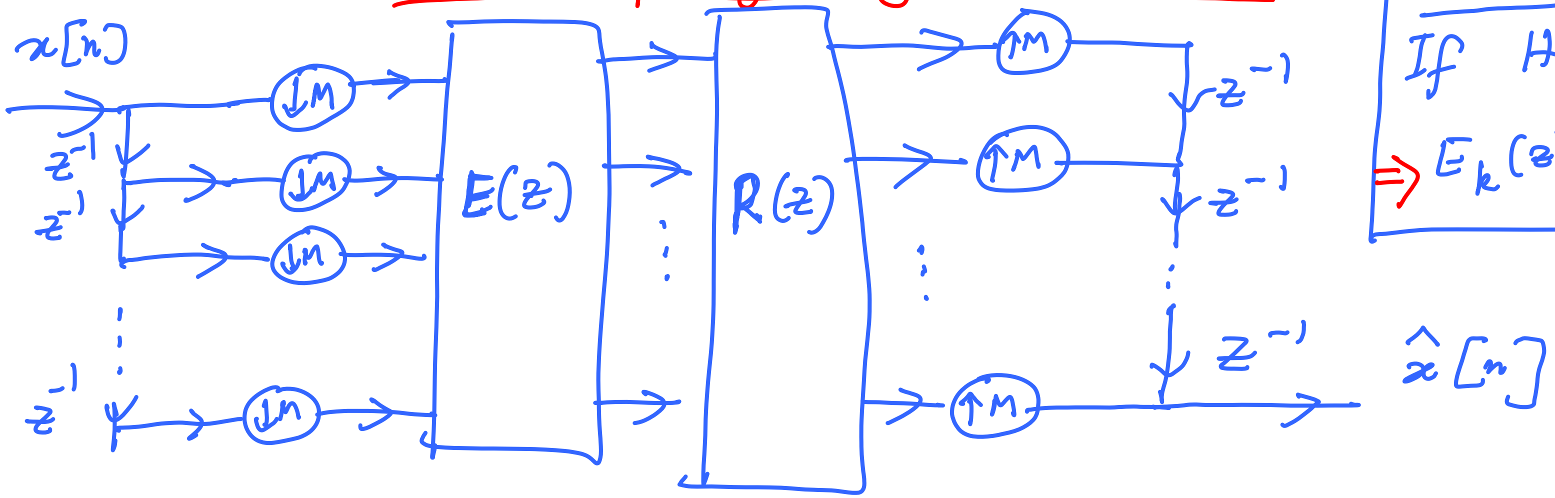
(Type 2 polyphase)



Architecture - I

Trick: Apply Noble Identities to simplify Arch. I further towards analysis on P. R. properties

A lot of Symmetry in Arch. II



Architecture - II

Aside:

If $H_k(z)$ is causal

$\Rightarrow E_k(z)$ is causal

$$P(z) = E(z)\,R(z)$$

If $P(z) = I$, we are left with a delay filter bank

GOAL: Choose $R(z)$ to meet the design specs (alias free, P.R.)

From our previous discussion on polyphase decomposition, we saw

$$P(z) = R(z)\,E(z)$$

If $P(z) = I \implies R(z) = E^{-1}(z)$

Theorem: (TASSP 1987) (Exercise: Go through the proof)

A necessary and sufficient condition for the P.R. property is $P(z)$ must take one of the following forms i.e.,

$$P(z) = d\begin{bmatrix} z^{-k} & 0 \\ 0 & z^{-k} \end{bmatrix} \quad \text{or} \quad P(z) = d\begin{bmatrix} 0 & z^{-k} \\ z^{-k-1} & 0 \end{bmatrix}$$

$$E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix} \qquad R(z) = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix}$$

$$R(z)\,E(z) = \begin{bmatrix} 2E_0(z)E_1(z) & 0 \\ 0 & 2E_0(z)E_1(z) \end{bmatrix}$$

$$= 2E_0(z)E_1(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, a system is PR iff $E_0(z)\,E_1(z)$ is a pure delay

If $H_0(z)$ is constrained to be a FIR, it is possible iff $E_0(z)$ and $E_1(z)$ are pure delays

$\Rightarrow$ $H_0(z)$ must be of the form

$$H_0(z) = c_0\, z^{-2n_0} + c_1\, z^{-(2n_1+1)}$$

If we need a FIR, we may have to give up on

$$H_1(z) = H_0(-z)$$

Idea :

For a general PR, we let $R(z) = d\, z^{-k} E^{-1}(z)$

a) If $E(z)$ is min ϕ, $R(z)$ hence $\{F_k(z)\}$ will be IIR but 'stable'

b) If somehow $\det(E(z))$ lands as a 'delay' we can hopefully construct FIR PR QMF banks

$\det(E(z)) = z^{-n_0} \implies$ Family of lossless matrices

This is however not a necessary condition for PR.

Example : Suppose

$$H(z) = \begin{bmatrix} 1+z^{-1} & 1-z^{-1} \\ 1-z^{-1} & 1+z^{-1} \end{bmatrix}$$

$$H(z)\,\tilde{H}(z) = c\,I \qquad (\text{Loss less})$$

$$H^{-1}(z) = \frac{1}{4}\begin{bmatrix} 1+z & 1-z \\ 1-z & 1+z \end{bmatrix}$$

Suppose $R(z) = c\,\underbrace{z^{-k}\,E^{-1}(z)}_{I}$. Consider $E(z) = H(z)$

We plug in $k=1$, $c=4$

$$R(z) = \begin{bmatrix} 1+z^{-1} & z^{-1}-1 \\ z^{-1}-1 & 1+z^{-1} \end{bmatrix}$$

$$E(z) = H(z)$$

$$\begin{aligned} H_0(z) &= E_{00}(z^2) + z^{-1} E_{01}(z^2) \\ &= 1 + z^{-2} + z^{-1}(1 - z^{-2}) \\ &= 1 + z^{-1} + z^{-2} - z^{-3} \end{aligned}$$

$$
\left.
\begin{aligned}
H_1(z) &= E_{10}(z^2) + z^{-1} E_{11}(z^2) \\
&= 1 - z^{-2} + z^{-1}\left(1 + z^{-2}\right) \\
&= 1 + z^{-1} - z^{-2} + z^{-3}
\end{aligned}
\right\}
$$

Analysis filters H_0, H_1 are FIR

$|||^{ly}$ for the synthesis bank,

$$
\left.
\begin{aligned}
F_0(z) &= z^{-1} R_{00}(z^2) + R_{10}(z^2) \\
F_1(z) &= z^{-1} R_{01}(z^2) + R_{11}(z^2) \\
F_0(z) &= z^{-1}\left(1 + z^{-2}\right) + z^{-2} - 1 = -1 + z^{-1} + z^{-2} + z^{-3} \\
F_1(z) &= 1 - z^{-1} + z^{-2} + z^{-3}
\end{aligned}
\right\}
$$

Synthesis filters are FIR

Special Filters and Properties

M^{th} band / Nyquist M filters

From polyphase decomposition,

$$H(z) = c + z^{-1} E_1(z^M) + z^{-2} E_2(z^M) + \dots + z^{-(M-1)} E_{M-1}(z^M)$$

$$x[n] \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$$\begin{aligned} Y(z) &= X(z^M)\, H(z) \\ &= c\, X(z^M) + \sum_{\ell=1}^{M-1} z^{-\ell} E_\ell(z^M)\, X(z^M) \end{aligned}$$

$y(Mn) = c\, x(n)$ by considering just the 1^{st} polyphase component!

*

Suppose $h(Mn) = \begin{cases} c & n = 0 \\ 0 & \text{else} \end{cases}$

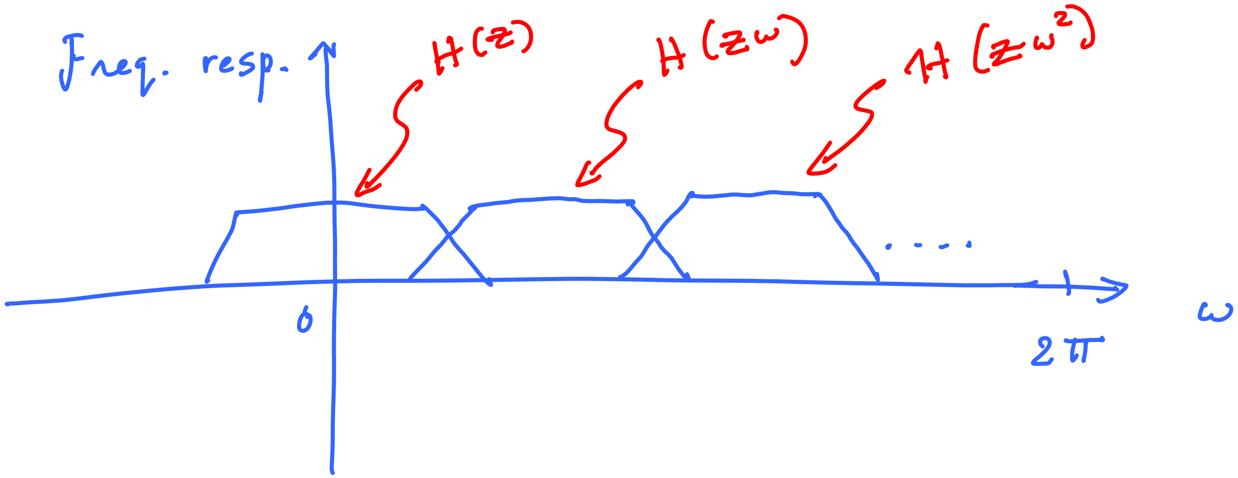
From the representation of $H(z)$ in the polyphase form,

$$\sum_{k=0}^{M-1} H(z\omega^k) = c + z^{-1}E_1(z^M) + \cdots + z^{-(M-1)}E_{M-1}(z^M)$$
$$+ \; c + z^{-1}\omega^{-1}E_1(z^M\omega^M) + \cdots + z^{-(M-1)}\omega^{-(M-1)}E_{M-1}(z^M\omega^M)$$
$$\vdots$$
$$+ \; c + z^{-1}\omega^{-(M-1)}E_1(z^M\omega^{(M-1)M}) + \cdots + z^{-(M-1)}\omega^{-(M-1)(M-1)} \cdot E_{M-1}(z^M\omega^{(M-1)M})$$

$$cM + z^{-1}\underbrace{\left[1 + \omega^{-1} + \cdots + \omega^{-(M-1)}\right]}_{}E_1(\cdot) + \cdots$$

0

If $c = \frac{1}{M} \Rightarrow \sum_{k=0}^{M-1} H(z\omega^k) = 1$



Ponder: Why are these called 'Nyquist' M filters?
What is the cross over frequency for each pair of adjacent freq. responses?

Half band filters

This is a special case of the Nyquist $M = 2$ filters.

$$H(z) = c + z^{-1} E_1(z^2)$$

$$h(2n) = \begin{cases} c & n = 0 \\ 0 & \text{else} \end{cases}$$

VERIFICATION

$$H(-z) = H(e^{j(\pi+\omega)})$$

$$H(e^{j\omega}) + H(e^{j(\pi+\omega)}) = 1$$

symmetric around $\pi/2$!

Examples:

$$H(z) = \begin{cases} 1 + z^{-3} \\ z + 1 + z^{-1} \\ 1 + z^{-1} + z^{-3} \end{cases}$$

$E_1(z) = z^{-1}$

$E_1(z) = 1 + z$ (anti causal)

$E_1(z) = 1 + z^{-1}$ (causal)

Filters have linear phase, but could be causal / anti causal!

NOTES

1) One can design good half band filters using mirror symmetric properties.

2) The generalization to Nyquist M filter design can be carried out through optimization techniques.

System Level Properties

A. Strictly Complementary Functions (SC)

$[H_0(z) \;\; H_1(z) \;\; \dots \;\; H_{M-1}(z)]$ are S.C. if their responses add to delay.

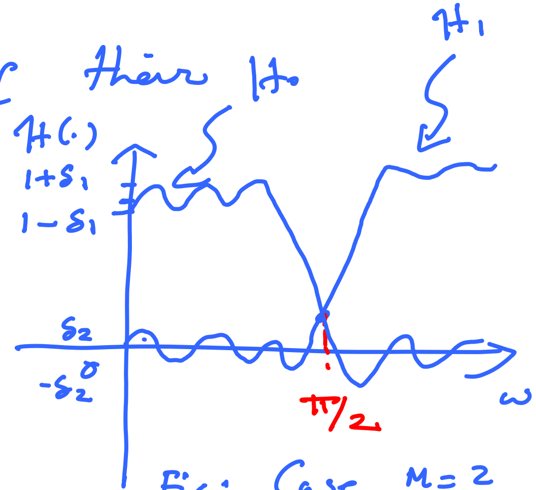


Fig: Case $M=2$
$c=1,\ n_0=\frac{N}{2}$

$$\sum_{k=0}^{M-1} H_k(z) = c\, z^{-n_0} \qquad c \neq 0$$

For $M = 2$ case, with $c = 1$ and $n_0 = N/2$,

$$H_0(e^{j\omega}) = e^{-j\omega N/2} H_R(\omega)$$

$$H_1(z) = z^{-N/2} - H_0(z)$$

$$\Rightarrow H_1(e^{j\omega}) = e^{-j\omega N/2}\left(1 - H_R(\omega)\right)$$

B. Power Complementary

$$\sum_{k=0}^{M-1} |H_k(e^{j\omega})|^2 = c \quad \forall \omega$$

$$\Rightarrow \underbrace{\sum_{k=0}^{M-1} H_k(e^{j\omega})\,\tilde{H}_k(e^{j\omega})}_{} = c$$

From P.R. perspective, one can choose $F_k(z) = \tilde{H}_k(z)$

To ensure causality, one can impose delay constraints into $\tilde{H}_k(z)$

so that the o/p is delayed

For $M = 2$, $c = 1$ PC

$$|H_1(e^{j\omega})|^2 = 1 - |H_0(e^{j\omega})|^2$$

$\therefore$ $H_1(e^{j\omega})$ is a spectral factor of $1 - |H_0(e^{j\omega})|^2$

C. **All Pass Complementary**

$$\sum_{k=0}^{M-1} H_k(z) = A(z)$$

where $A(z)$ is an all pass filter

S.C $\Longrightarrow$ A.P.C, but not "otherwise"

$\not\Longleftarrow$

Reconstruction is free from 'amplitude distortion'

D. <u>Doubly Complementary</u>

$\{H_k(z)\}_{k=0}^{M-1}$ satisfying both P. C. & A. P. C.

are double complementary